# HAUSDORFF DIMENSION OF SOME INVARIANT SETS

BY

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Abstract of Dissertation Presented to the Graduate School of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

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We study the Hausdorff dimension for some invariant sets. For a hyperbolic set of a diffeomorphism or a flow of diffeomorphisms, the uniform stable and unstable Lyapunov exponents have been defined. Using the uniform Lyapunov exponents we define a characteristic function. At some real positive number $t$ the characteristic function equals the value of the topological entropy of the diffeomorphism on the hyperbolic set. We prove that $t$ is an upper bound for the Hausdorff dimension of this hyperbolic set. Technically, we use a pinching condition to cope with the nonlinearity of the diffeomorphism, to prove that the iterated image of a ball is somehow contained in the image of the ball under the derivative of the iterated diffeomorphism. Some earlier results are thus improved. In Chapter 4 we have also given upper bounds for the Hausdorff dimension of the transverse of a hyperbolic set with stable and unstable manifolds, using the topological pressure on the hyperbolic set.

Chapter 5 deals with the self similar sets. The self similar sets of iterated mapping systems have been studied thoroughly in the cases where construction diffeomorphisms are conformal. We study the case where the construction diffeomorphisms are not necessarily conformal. We give a new distortion lemma for the construction mappings. Using topological pressure on a shift space we give an upper estimate of the Hausdorff dimension, when the construction diffeomorphisms are $C^{1+\kappa}$ and satisfy a $\kappa$ pinching condition for some $\kappa \leq 1$. Moreover, if the construction diffeomorphisms also satisfy the disjoint open set condition we then give a lower bound for the Hausdorff dimension. We also obtain the continuity of the Hausdorff dimension in the $C^1$ topology at conformal constructions.

We study the image of a disc under the iteration of a Hénon map, which enables us to find an upper bound for the Hausdorff dimension and capacity of the Hénon attractor. That improves some earlier estimates when applied to the case of a Hénon attractor.

# CHAPTER 1

# INTRODUCTION

For a topological space, especially a subset of an Euclidean space, a variety of "dimensions" have been introduced and studied. Among them it has been widely agreed that "Hausdorff dimension" is the most important one to use. In the case of Euclidean spaces, the Hausdorff dimension agrees with their topological dimension which are integers. Those subsets with non-integer Hausdorff dimensions are called "fractals." The sets we study here, i.e., the hyperbolic sets, the self similar sets, and the Hénon attractors are mostly fractals. Understanding their structures, in particular finding their dimensions, is one of the interesting open problems.

In 1914, Carathéodory introduced the classical idea of using covers of sets to define measures. That idea was adopted by Hausdorff in 1919 to define a special measure which is now called a Hausdorff measure for his study of the Cantor set. This special measure can be defined for any dimension. However, it either vanishes or equals infinity in all but at most one dimension, which is now called the Hausdorff dimension. In his 1919 paper Hausdorff found the dimension of the Cantor set is $\log_3 2$, with a positive Hausdorff measure at that dimension, see Hausdorff (1919). The properties of Hausdorff measure and Hausdorff dimension have been thoroughly studied by Besicovitch et al.

In this paper, we study the Hausdorff dimension of the hyperbolic sets, the self similar sets, and the Hénon attractors.

Generalizing the concept of a hyperbolic fixed point for a differentiable self map of a smooth manifold, people introduced the hyperbolic sets. For convenience, we call a differentiable one to one self map of a smooth manifold a diffeomorphism. The hyperbolic sets have been studied by Smale (1967), Mather (1968), Hirsch and Pugh (1970) among other dynamists. After the introduction of Pesin's theory, it was realized that Hausdorff dimension of invariant measures was related to dynamics, and the relationship was studied in its final form by Ledrappier and Young (1985). It is still an open problem to find the Hausdorff dimension of an invariant set. Fathi (1989) gave an upper bound for the Hausdorff dimension of the hyperbolic sets, using the topological entropy. His result shows that the Hausdorff dimension can be upper bounded by a fraction of the topological entropy over the skewness of the hyperbolic set. We use Fathi's idea. Technically, we use a pinching condition to cope with the nonlinearity of the diffeomorphism, to prove that the iterated image of a ball is somehow contained in the image of the ball under the derivative of the iterated diffeomorphism. For a hyperbolic set of a diffeomorphism or a flow of diffeomorphisms, the uniform stable and unstable Lyapunov exponents have been defined. Using the uniform Lyapunov exponents we define a characteristic function. At some real positive number $t$ the characteristic function equals the value of the topological entropy of the diffeomorphism on the hyperbolic set. We prove that $t$ is an upper bound for the Hausdorff dimension of this hyperbolic set. Some earlier results are thus improved. Moreover, under the pinching condition, we use the topological pressure to upper bound the Hausdorff dimensions of a hyperbolic set and the transversals of the hyperbolic set with stable and unstable manifolds. For the two-dimensional diffeomorphism with a horeseshoe as a basic set, our upper bounds for the transversals agree with the identities given by McCluskey

and Manning (1983). However, due to the increase of dimensions, one does not expect as explicit an identity for the Hausdorff dimension as the one given by McCluskey and Manning (1983).

Self similar sets of iterated mapping systems are a special kind of limit sets. They have been studied thoroughly in the cases where construction diffeomorphisms are conformal. Essentially, the conformal constructions are one-dimensional and one has a simple formula for the Hausdorff dimension. We study the case where the construction diffeomorphisms are not necessarily conformal. We give a new version of distortion lemma. Using topological pressure on a shift space we give an upper estimate of the Hausdorff dimension, when the construction diffeomorphisms are $C^{1+\kappa}$ and satisfy a $\kappa$ pinching condition for some $\kappa \leq 1$. Moreover, if the construction diffeomorphisms also satisfy the disjoint open set condition we then give a lower bound for the Hausdorff dimension. We also obtain the continuity of the Hausdorff dimension in the $C^1$ topology at conformal constructions.

We study the image of a disc under the iteration of a Hénon map, which enables us to find an upper bound for the Hausdorff dimension and capacity of the Hénon attractor. That improves some earlier estimates when applied to the case of a Hénon attractor.

Our main results are the following four theorems. For the upper bound of Hausdorff dimension of a hyperbolic set using uniform Lyapunov exponents, we have

THEOREM I. *Let $f \colon U \to X$ be a $C^2$ diffeomorphism with a compact hyperbolic set $K \subset U$, where $U$ is an open subset in the $n$-dimensional Riemannian manifold $X$. Let $\mu_1^s, \mu_2^s, \cdots, \mu_{n_1}^s$ be the stable uniform Lyapunov exponents of $f$; and let*

*$\mu_1^u, \mu_2^u, \cdots, \mu_{n_2}^u$ be the unstable uniform Lyapunov exponents of $f$. Relabel the numbers $\mu_1^s, \cdots, \mu_{n_1}^s$ and $\mu_1^u, \cdots, \mu_{n_2}^u$ as $\mu_1, \cdots, \mu_n$ with the order*

$$0 > \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.$$

*Define $\tau : [0, n] \to R$ the characteristic function by*

$$\tau(s) = \mu_1 + \cdots + \mu_{[s]} + (s - [s])\mu_{[s]+1}.$$

*Let $\lambda = ent(f\,|_K)$ be the topological entropy of $f$ on $K$. Let $D = \tau^{-1}(-2\lambda)$, or $D = n$ if $\mu_1 + \mu_2 + \cdots + \mu_n > -2\lambda$. If $AB^2 < 1$, where $A = \lim A_k$, $B = \lim B_k$, and*

$$A_k = \max\{\max_{x \in K} \|T_x f^k\|^{\frac{1}{k}}, \max_{x \in K} \|T_x f^{-k}\|^{\frac{1}{k}}\},$$
$$B_k = \max\{\max_{x \in K} \|T_x f^k\,|_{E^s}\|^{\frac{1}{k}}, \max_{x \in K} \|T_x f^{-k}\,|_{E^u}\|^{\frac{1}{k}}\},$$

*then the Hausdorff dimension $HD(K) \leq D$.*

Theorem I is proved in Chapter 3. For the upper bound of Hausdorff dimension of a $C^2$ hyperbolic set and its transverse using topological pressure we have

**THEOREM II.** *If $f$ is $C^2$, pinched, then $HD(K) \leq t$, where $t$ is uniquely such that*

$$\max\{P(f|_K, \lambda_{t_1}^s) + P(f|_K, \lambda_{t_2}^u) = 0 |\; t_1 + t_2 = t\} = 0.$$

*If $f$ is $C^2$, pinched, and $P(f|_K, \lambda_t^s) = 0$, then*

$$HD(K \cap W^s(x, f)) \leq t.$$

*If $f$ is $C^2$, pinched, and $P(f|_K, \lambda_t^u) = 0$, then*

$$HD(K \cap W^u(x, f)) \le t.$$

Theorem II is proved in Chapter 4. For the self similar sets, we give upper and lower bounds for the Hausdorff dimension. We have

THEOREM III. *Let $\{\varphi_{ij}\}$ be the $C^1$ construction diffeomorphisms for the self similar set $E$, satisfying the $\kappa$ pinching condition for some positive number $\kappa \le 1$. Suppose the derivatives of all $\{\varphi_{ij}\}$ are Hölder continuous of order $\kappa$. If $t$ is the unique positive number such that the topological pressure $P(\sigma, \lambda_t) = 0$, then the Hausdorff dimension $HD(E) \le t$.*

*Let $\{\varphi_{ij}\}$ be the $C^1$ construction diffeomorphisms for the self similar set $E$ in an Euclidean space $R^l$, satisfying both the $\kappa$ pinching condition for some positive number $\kappa \le 1$, and in addition the disjoint open set condition. Suppose the derivatives of all $\{\varphi_{ij}\}$ are Hölder continuous of order $\kappa$. If $t$ is the unique positive number such that the topological pressure $P(\sigma, \lambda_t) = 0$, then the Hausdorff dimension*

$$HD(E) \ge \frac{t}{1+\kappa} - l\kappa.$$

Theorem III is proved in Chapter 5. For a Hénon attractor, we have

THEOREM IV. *Let $0 < b < 1/2$. Suppose that $\Lambda$ is a compact invariant attractor under the Hénon map $H_{a,b}$ and an orbit $\{z_n = (x_n, y_n) = H^n(z_0)\}$ is dense in $\Lambda$. Let $R$ be the bound for $\Lambda$ such that for every $(x, y) \in \Lambda$ we have $|x|, |y| \le R$. Denote $m = \dfrac{2R+1}{b}$, and*

$$A = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(b + \frac{2|x_i|}{m} + \frac{1}{m^2}) < 0.$$

*Then the upper capacity*

$$\bar{C}(\Lambda) \leq 2 + \frac{A}{\log m} < 2.$$

Theorem IV is proved in Chapter 6. Finally, we conclude this work by discussing the difficulty of relating the Hausdorff dimension of the Hénon attractor to the topological entropy of the Hénon map.

# CHAPTER 2

# HAUSDORFF DIMENSION AND CAPACITY

## §2.1. Hausdorff Measure and Dimension

Let $(X, d)$ be a metric space and let $s$ be any non-negative real number. Let $K \subset X$ be a subspace of $X$. Let $\alpha$ be a collection of balls, with the union of all balls containing $K$. Such a collection $\alpha$ is called a covering of $K$. For any real number $\varepsilon > 0$, if every ball in $\alpha$ has radius less than or equal to $\varepsilon$, then $\alpha$ is called an $\varepsilon$ covering of $K$. Suppose $\alpha = \{B_i : i \in I\}$ and each ball $B_i$ has a radius $r_i$. We denote by $\mathcal{H}^s_\varepsilon(K)$ the quantity

$$\inf_\alpha \sum_{i \in I} r_i^s$$

where the infimum is taken over all $\varepsilon$ coverings of $K$. If another real positive number $\delta < \varepsilon$, then any $\delta$ covering is also an $\varepsilon$ covering. Thus $\mathcal{H}^s_\varepsilon(K)$ is no larger than $\mathcal{H}^s_\delta(K)$. One can define the quantity

$$\mathcal{H}^s(K) = \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(K) = \sup_{\varepsilon > 0} \mathcal{H}^s_\varepsilon(K).$$

We shall call the quantity $\mathcal{H}^s(K)$ the Hausdorff $s$ dimensional measure of $K$. It can be easily verified that Hausdorff measure $\mathcal{H}^s$ is an outer measure with the

following properties:

$$\mathcal{H}^s(\emptyset) = 0,$$
$$\mathcal{H}^s(\cup_n A_n) \leq \sum_n \mathcal{H}^s(A_n),$$
$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B),$$

where $A$ and $B$ have disjoint closures. Moreover, for any real positive numbers $s, t, \varepsilon$, by the definition,

$$\begin{aligned}\mathcal{H}^t_\varepsilon(K) &= \inf_{\varepsilon \text{ coverings}} \sum_{i \in I} r_i^t \\ &\leq \varepsilon^{t-s} \inf_{\varepsilon \text{ coverings}} \sum_{i \in I} r_i^s \\ &= \varepsilon^{t-s} \mathcal{H}^s_\varepsilon(K).\end{aligned}$$

Therefore, if $s < t$ and $\mathcal{H}^s(K) < +\infty$, then

$$\mathcal{H}^t(K) \leq \lim_{\varepsilon \to 0} \varepsilon^{t-s} \mathcal{H}^s_\varepsilon(K) = 0 \cdot \mathcal{H}^s(K) = 0.$$

This also implies that if $t < s$ and $\mathcal{H}^s(K) > 0$, then

$$\mathcal{H}^t(K) = +\infty.$$

Hence we have the following identity:

$$\inf\{s : \mathcal{H}^s(K) = 0\} = \sup\{s : \mathcal{H}^s(K) = \infty\}.$$

The Hausdorff dimension of the set $K$ is defined by the above number, denoted by $HD(K)$. It is worth to notice that for an Euclidean space the Hausdorff

dimension equals the topological dimension, and the Hausdorff measure is equal to the Lebesgue measure multiplied by a constant factor. Also, it is noticed that many subsets of Euclidean spaces have non-integer Hausdorff dimensions. Such sets are called fractals.

The Hausdorff measure was first introduced by Hausdorff in 1919 based on Carathéodory's idea of using covers to define measures. Since then many research works have been done. Here we give a well-known Lemma of Frostman which will be used later when estimating the Hausdorff dimension of the self similar sets. See Temam (1988) or Hurewicz and Wallman (1948) for other properties of the Hausdorff dimensions and Hausdorff measures, and examples of fractal sets.

## §2.2. Frostman's Lemma

The following lemma, Frostman's lemma, is widely used when estimating Hausdorff dimensions. It gives a necessary and sufficient condition for a subset of an Euclidean space to allow a non trivial Hausdorff measure. For convenience we use $|B|$ to denote the diameter of a set $B$. If $d$ is the metric then

$$|B| = \sup\{d(x,y) : x, y \in B\}.$$

LEMMA 2.2.1. *Let $(X, d)$ be an $l$ dimensional Euclidean space or an $l$ dimensional manifold, with the metric denoted by $d$. Suppose $K \subset X$ is a compact subset. Then $K$ has a positive $s$ dimensional Hausdorff measure if and only if there can be defined a probability measure $\mu$ on $K$, with the property that for all balls $B$ in $(K, d)$ and some positive constant $C$, the measure satisfies*

$$\mu(B) \leq C \cdot |B|^s.$$

**Proof**: The "if" part is obvious. Given a probability measure $\mu$ on $K$ with

$$\mu(B) \leq C \cdot |B|^s$$

for all balls $B$, we see that probability measure is close to a Hausdorff measure. In fact, for any given $\varepsilon$ covering $\{B_i : i \in I\}$ where the radius $r_i = |B_i|/2$, we have

$$\sum_{i\in I} r_i^s \geq \sum_{i\in I} |B_i|^s/2^s \geq \sum_{i\in I} \mu(B_i)/2^s C \geq \mu(K)/2^s C = 1/2^s C.$$

Thus the Hausdorff measure $\mathcal{H}^s(K) \geq 1/2^s C > 0$. That proves the "if" part.

To prove the "only if" part, let $s > 0$ be given such that $\mathcal{H}^s(K) > 0$, and let $\{B_i : i \in I\}$ be any covering of balls. For any $\varepsilon > 0$, one must have either $\sup\{|B_i| : i \in I\} \leq \varepsilon$ or $\sup\{|B_i| : i \in I\} > \varepsilon$. Thus it follows that

$$\sum_{i\in I} |B_i|^s \geq \gamma, \text{ where } \gamma = \min\{\varepsilon^s, \mathcal{H}^s_\varepsilon(K)\}.$$

If $\varepsilon$ is small enough, then $H^s_\varepsilon(K) > 0$, and also $\gamma$ will be strictly positive. Therefore we can fix some $\gamma > 0$ such that for all coverings of balls of $K$, the sum

$$\sum_{i\in I} |B_i|^s \geq \gamma.$$

The same conclusion will be true if the balls are replaced by cubes.

For a positive integer $m$ we define a dyadic interval of order $m$ to be any interval of the form $[p2^{-m}, (p+1)2^{-m}]$, where $p$ is any integer. A dyadic cube of order $m$ is defined as a Cartesian product of dyadic intervals of order $m$. Consider now a covering of $K$ using dyadic cubes of order $m$. In order to introduce a probability measure on $K$, let us distribute the mass $2^{-ms}$ uniformly on each

dyadic cube of order $m$ that intersects $K$. We then get a measure $\mu_m^m$. On each dyadic cube of order $m-1$ we keep the measure $\mu_m^m$ if its mass does not exceed $2^{-s(m-1)}$, or we multiply $\mu_m^m$ by a constant less than 1 in such a way that the resulting measure has mass $2^{-s(m-1)}$; that way we get a measure $\mu_{m-1}^m$. For each $k \geq 1$, the measure $\mu_{m-k-1}^m$ is obtained from $\mu_{m-k}^m$ in the same way, that is

$$\mu_{m-k-1}^m = \lambda(D) \cdot \mu_{m-k}^m$$

on each dyadic cube $D$ of order $m-k-1$, with

$$\lambda(D) = \min\{1, 2^{-s(m-k-1)}\mu_{m-k}^m(D)^{-1}\}.$$

Clearly $\mu_{m-k}^m$ is independent of $k$ when $k$ is large enough. Then we can write

$$\mu_{m-k}^m = \mu^m.$$

Notice the measure $\mu^m$ is carried by the dyadic cubes of order $m$ which intersect $K$ and we have

$$\mu^m(D) \leq 2^{-s(m-k)}$$

for each dyadic cube $D$ of order $m-k$ where $k = 0, 1, \cdots$. Each point of $K$ is contained in some dyadic cube $D$ such that the following equality holds,

$$\mu^m(D) = c|D|^s, \; c = l^{-s/2},$$

where $l$ is the topological dimension of $X$. For each point of $K$ let us consider the largest such cube $D$: we get a covering of $K$ by disjoint dyadic cubes, $D_n$.

Then using the property of $\gamma$, one has the following estimate for the norm of the measure $\mu^m$:

$$\|\mu^m\| = \sum_n \mu^m(D_n) = c\sum_n |D_n|^s \geq c\gamma.$$

We normalize the measure $\mu^m$ through $\mu^m/\|\mu^m\|$, still denoted as $\mu^m$. We thus get a probability measure $\mu^m$ with

$$\mu^m(D) \leq \gamma^{-1}|D|^s$$

for each dyadic cube $D$ of order less than or equal to $m$, and $\mu^m$ is carried by a given neighborhood of $K$ if $m$ is large enough. From the sequence $\{\mu^m, m = 1, 2, \cdots\}$ we can extract a subsequence $\mu^{m_j}$ which converges weakly. The limit, denoted by $\mu$, is a probability measure carried by $K$ and

$$\mu(D) \leq \gamma^{-1}|D|^s$$

for each dyadic cube $D$.

Now, given an interval of length $b$, find $k$ such that $2^{-k-1} \leq b < 2^{-k}$; then the interval is covered by two dyadic intervals of length $2^{-k}$. Given a ball of diameter $b$, it is covered by $2^l$ dyadic cubes of order $k$. Therefore,

$$\mu(B) \leq 2^l\gamma^{-1}|B|^s$$

for each ball $B$. Let $C = 2^l\gamma^{-1}$ and then the "only if" part is complete. ∎

## §2.3. The Capacity

There have been many other dimensions defined for a topological space besides the Hausdorff dimension. Among them is the capacity which some people call fractal dimension and others call entropy dimension. Let $(X, d)$ be any metric space. Let $N(X, \varepsilon)$ denote the minimal cardinality of an $\varepsilon$ covering of balls of $X$. Note that $N(X, \varepsilon)$ is the minimum number of radii $\varepsilon$ balls needed to cover $X$. When $X$ is given, $N(X, \varepsilon)$ is a decreasing function of $\varepsilon$. The capacity of $X$, denoted by $C(X)$, is given by

$$C(X) = \limsup_{\varepsilon \to 0} \frac{\log N(X, \varepsilon)}{-\log \varepsilon}.$$

The capacity $C(X)$ is also called the upper capacity. In the same way we define the lower capacity $\underline{C}(X)$ by

$$\underline{C}(X) = \liminf_{\varepsilon \to 0} \frac{\log N(X, \varepsilon)}{-\log \varepsilon}.$$

Mandelbrot has suggested an alternative way of defining capacity. Let

$$\mathcal{C}^s(X) = \limsup_{\varepsilon \to 0} \varepsilon^s N(X, \varepsilon).$$

The upper capacity is given by

$$C(X) = \inf\{s > 0 : \mathcal{C}^s(X) = 0\}.$$

As for lower capacity, if we let

$$\underline{\mathcal{C}}^s(X) = \liminf_{\varepsilon \to 0} \varepsilon^s N(X, \varepsilon).$$

Then the lower capacity is given by

$$\underline{C}(X) = \inf\{s > 0 : \underline{C}^s(X) = 0\}.$$

It is easy to verify the two definitions for upper and lower capacity agree with each other. A covering using diameter $\varepsilon$ balls is indeed an $\frac{1}{2}\varepsilon$ covering. So

$$\begin{aligned}\underline{C}^s(X) &= \liminf_{\varepsilon\to 0} \varepsilon^s N(X,\varepsilon)\\ &\geq \lim_{\varepsilon\to 0} \inf_{\varepsilon \text{ coverings}} \sum_{i\in I} r_i^s\\ &= \mathcal{H}^s(X).\end{aligned}$$

We have

$$HD(X) \leq \underline{C}(X) \leq C(X).$$

It is worth mentioning that the inequality can indeed be a strict one. See Temam (1988) for such examples. The following fact due to Fathi (1988) about capacity is very useful. It rewrites the capacities as the limits of sequences.

LEMMA 2.3.1. *If $0 < \theta < 1$ and $\delta > 0$, then*

$$C(X) = \limsup_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{-\log\varepsilon} = \limsup_{n\to\infty} \frac{\log N(X,\theta^n\delta)}{-n\log\theta},$$

$$\underline{C}(X) = \liminf_{\varepsilon\to 0} \frac{\log N(X,\varepsilon)}{-\log\varepsilon} = \liminf_{n\to\infty} \frac{\log N(X,\theta^n\delta)}{-n\log\theta}.$$

PROOF: To simplify notations let $\delta = 1$ and define

$$n(\varepsilon) = [\log\varepsilon/\log\theta].$$

We have

$$\lim_{\varepsilon \to 0} n(\varepsilon) = \infty, \text{ and } \lim_{\varepsilon \to 0} \frac{\log \varepsilon}{\log \theta^{n(\varepsilon)}} = 1.$$

Since $N(X, \varepsilon)$ decreasing of $\varepsilon$,

$$N(X, \theta^{n(\varepsilon)}) \leq N(X, \varepsilon) \leq N(X, \theta^{n(\varepsilon)+1}).$$

It is reasonable to assume $\varepsilon < 1$. We have

$$\frac{\log N(X, \theta^{n(\varepsilon)})}{-\log \varepsilon} \leq \frac{\log N(X, \varepsilon)}{-\log \varepsilon} \leq \frac{\log N(X, \theta^{n(\varepsilon)+1})}{-\log \varepsilon}.$$

Now let $\varepsilon \to 0$, and we can obtain the equalities desired. ∎

For other properties of capacity, see Temam (1988).

## CHAPTER 3

## THE HYPERBOLIC SET AND LYAPUNOV EXPONENTS

### §3.1. The Hyperbolic Set

In this chapter we give an upper bound for the Hausdorff dimension of a hyperbolic set using the uniform Lyapunov exponents. By diffeomorphism we mean a differentiable one to one map. Let $U$ be an open set in a smooth Riemannian manifold $X$ and $f : U \to X$ a $C^1$ diffeomorphism from $U$ onto an open subset of $X$. If $x$ is in $X$, we denote by $T_xX$ the tangent space of $X$ at $x$, and by $T_xf : T_xX \to T_{f(x)}X$ the derivative of $f$ at $x$. More generally we denote by $T_KX$ the restriction of the tangent bundle of $X$ to a subset $K$ of $X$.

DEFINITION 3.1.1. *We say that a compact set $K \subset U$ is hyperbolic for $f$ if $f(K) = K$ and if there is a $Tf$ invariant splitting of the tangent bundle $E = T_KX = E^s \oplus E^u$ such that*

$$\lim_{k\to+\infty} \frac{1}{k} \log(\max_{x\in K} \|T_xf^k|_{E^s}\|) < 0,$$

*and*

$$\lim_{k\to+\infty} \frac{1}{k} \log(\max_{x\in K} \|T_xf^{-k}|_{E^u}\|) < 0,$$

*where the norm is obtained from a Riemannian metric on $X$.*

In the above definition we used the subadditivity of the two sequences

$$\log(\max_{x\in K} \|T_xf^{-k}|_{E^u}\|), \text{ and } \log(\max_{x\in K} \|T_xf^k|_{E^s}\|)$$

to obtain the existence of the limits. The number $\tau$ given by

$$\tau = \max\{\lim_{k\to+\infty}\max_{x\in K}\|T_xf^k\mid_{E^s}\|^{\frac{1}{k}}, \lim_{k\to+\infty}\max_{x\in K}\|T_xf^{-k}\mid_{E^u}\|^{\frac{1}{k}}\},$$

is called the skewness of $K$. Note that the notion of hyperbolicity does not depend on the choice of the Riemannian metric. We will need the notion of topological entropy, which is defined in the following section. Refer to Walters (1982) for different definitions and properties. The definition of the Hausdorff dimension is in Chapter 2 and the theory of Hausdorff dimension can be found in Hurewicz and Wallman (1948). One can also use Temam (1988) as a reference for the definition and properties of the Hausdorff dimension of a metric space. The following theorem gives a better upper bound for Hausdorff dimension than the one obtained by Fathi (1989), also see Fathi (1988) for similar results in the case of hyperbolic linear toral maps. For related results in dimension 2, see McCluskey and Manning (1983); and see also Ledrappier and Young (1985) for Hausdorff dimension of invariant measures. Some earlier related works can be found in Ledrappier (1981) and Frederickson, Kaplan, York and York (1983). For the definition of Lyapunov exponents, see Section 3.3.

THEOREM I. *Let $f\colon U \to X$ be a $C^2$ diffeomorphism with a compact hyperbolic set $K \subset U$, where $U$ is an open subset in the $n$-dimensional Riemannian manifold $X$. Let $\mu_1^s, \mu_2^s, \cdots, \mu_{n_1}^s$ be the stable uniform Lyapunov exponents of $f$; and let $\mu_1^u, \mu_2^u, \cdots, \mu_{n_2}^u$ be the unstable uniform Lyapunov exponents of $f$. Relabel the numbers $\mu_1^s, \cdots, \mu_{n_1}^s$ and $\mu_1^u, \cdots, \mu_{n_2}^u$ as $\mu_1, \cdots, \mu_n$ with the order*

$$0 > \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.$$

*Define $\tau : [0,n] \to R$ the characteristic function by*

$$\tau(s) = \mu_1 + \cdots + \mu_{[s]} + (s - [s])\mu_{[s]+1}.$$

*Let $\lambda = ent(f\mid_K)$ be the topological entropy of $f$ on $K$. Let $D = \tau^{-1}(-2\lambda)$, or $D = n$ if $\mu_1 + \mu_2 + \cdots + \mu_n > -2\lambda$. If $AB^2 < 1$, where $A = \lim A_k$, $B = \lim B_k$, and*

$$\begin{aligned} A_k &= \max\{\max_{x\in K} \|T_x f^k\|^{\frac{1}{k}}, \max_{x\in K} \|T_x f^{-k}\|^{\frac{1}{k}}\},\\ B_k &= \max\{\max_{x\in K} \|T_x f^k\mid_{E^s}\|^{\frac{1}{k}}, \max_{x\in K} \|T_x f^{-k}\mid_{E^u}\|^{\frac{1}{k}}\},\end{aligned}$$

*then the Hausdorff dimension $HD(K) \le D$.*

The condition $AB^2 < 1$ is called the pinching condition.

REMARK 1: In fact, after reordering, $B = \exp \mu_1$. So $B$ is the largest of these Lyapunov numbers. Neither $A$ nor $B$ depends on the choice of the Riemannian metric. Some people would prefer to define $A = A_1$, $B = B_1$, as Hirsch and Pugh (1970). In the case when $K = X$, the condition $AB^2 < 1$ implies that the stable and unstable foliations of $f$ are $C^1$, see Hirsch and Pugh (1970). Remark also that the condition $AB^2 < 1$ is open in the space of $C^1$ diffeomorphisms with $C^1$ topology.

REMARK 2: We choose sets of the form $\cap_{i=-m}^{i=m} f^{-i}B(f^i(x),\alpha)$ to cover $K$ where $x$ is in a $(2m+1,\alpha)$ spanning subset (of $K$) whose cardinality grows according to $2\lambda$. The lifting of $\cap_{i=-m}^{i=m} f^{-i}B(f^i(x),\alpha)$ to the tangent space $T_xX$ looks roughly like an ellipsoid whose axes are given by the uniform Lyapunov exponents, and now the Hausdorff measure of $K$ can be estimated by covering these ellipsoids by balls. The pinching condition $AB^2 < 1$ is needed in Lemma 3.4.2 to cope with the effects of nonlinearity on these ellipsoids.

## §3.2. The Topological Entropy

Let $X$ be a compact topological space and let $f : X \to X$ be a continuous self map on $X$. We shall use open covers of $X$, denoted by $\alpha$, $\beta, \cdots$, to define the topological entropy of $f$ which we denote by $ent(f)$.

If $\alpha$, $\beta$ are open covers of $X$, then denote by $\alpha \vee \beta$ their join, which is the open cover by all sets of the form $A \cap B$ where $A \in \alpha, B \in \beta$. The join of any finite collection of open covers of $X$ is defined similarly. For the open cover $\alpha$, let $N(\alpha)$ denote the number of open sets in $\alpha$, and let $H(\alpha) = \log N(\alpha)$. Since $f$ is continuous $f^{-1}\alpha$ is also an open cover, with $N(f^{-1}\alpha) \leq N(\alpha)$. Thus we can introduce the open cover $\vee_{i=0}^{n-1} f^{-i}\alpha$ and let $a_n = H(\vee_{i=0}^{n-1} f^{-i}\alpha)$.

Now $N(\alpha \vee \beta) \leq N(\alpha) \cdot N(\beta)$, and hence $H(\alpha \vee \beta) \leq H(\alpha) + H(\beta)$. Thus the sequence $\{a_n\}$ has the subadditivity as follows:

$$\begin{aligned} a_{k+n} &= H(\vee_{i=0}^{k+n-1} f^{-i}\alpha) \leq H(\vee_{i=0}^{k-1} f^{-i}\alpha) + H(f^{-k} \vee_{i=0}^{n-1} f^{-i}\alpha) \\ &\leq a_k + a_n. \end{aligned}$$

The subadditivity implies that

$$\lim_{n\to\infty} \frac{1}{n} a_n = \lim_{n\to\infty} \frac{1}{n} H(f^{-k} \vee_{i=0}^{n-1} f^{-i}\alpha)$$

exists. We call it the entropy of $f$ relative to the cover $\alpha$, denoted by $h(f, \alpha)$. For the continuous self map $f$ of $X$, the topological entropy is given by

$$ent(f) = \sup_{\alpha} h(f, \alpha)$$

where the supremum is taken over all the open covers of $X$.

For $(X, d)$ a metric space, Bowen introduced the following alternative way of defining the topological entropy. Bowen's definition relates the topological entropy to the spanning set. This is used by Fathi (1989) to relate the upper bound of the Hausdorff dimension of a hyperbolic set to the topological entropy of the diffeomorphism. We also use this definition when relating the Lyapunov exponents and topological entropy to the Hausdorff dimension. See Walters (1982) for a proof that in case of a metric space Bowen's definition is equivalent to the one using covers given above.

For each positive integer define a new metric on $X$ by

$$d_n(x, y) = \max_{0 \le i \le n-1} d(f^i(x), f^i(y)),$$

where $x, y \in X$. Let $K$ be a compact subset of $X$ and let $F \subset K$. For a real number $\varepsilon > 0$ we say that $F$ is an $(n, \varepsilon)$ spanning set of $K$ if for every point $x \in K$ there exists $y \in F$ such that $d_n(x, y) \le \varepsilon$. A subset $E$ of $K$ is said to be $(n, \varepsilon)$ separated if $d_n(x, y) > \varepsilon$ for any distinct $x, y \in E$. Denote the smallest cardinality of any $(n, \varepsilon)$ spanning set by $r_n(\varepsilon, K, f)$; and denote the largest cardinality of any $(n, \varepsilon)$ separated set by $s_n(\varepsilon, K, f)$. It is easy to check that

$$r_n(\varepsilon, K, f) \le s_n(\varepsilon, K, f) \le r_n(\varepsilon/2, K, f).$$

Now let

$$r(\varepsilon, K, f) = \limsup_{n \to \infty} r_n(\varepsilon, K, f),$$

and

$$s(\varepsilon, K, f) = \limsup_{n \to \infty} s_n(\varepsilon, K, f).$$

Notice that $r(\varepsilon, K, f)$ and $s(\varepsilon, K, f)$ have a common limit when $\varepsilon \to 0$. The topological entropy of $f$, denoted by $ent(f)$, is given by the common limit:

$$ent(f) = \sup_{K \subset X} \lim_{\varepsilon \to 0} r(\varepsilon, K, f) = \sup_{K \subset X} \lim_{\varepsilon \to 0} s(\varepsilon, K, f),$$

where the supremum is taken over the collection of all compact subsets of $X$.

## §3.3. The Uniform Lyapunov Exponents

Let $L : E \to E'$ be a linear map between $n$-dimensional Euclidean spaces. We write

$$\alpha_i(L) = \sup_{F \subset E, dim F = i} \inf_{\varphi \in F, \|\varphi\|_E = 1} \|L(\varphi)\|_{E'},$$

and $\omega_i(L) = \alpha_1(L) \cdots \alpha_i(L)$ for $i = 1, 2, \cdots, n$. One can verify that

$$\alpha_1(L) \geq \alpha_2(L) \geq \cdots \geq \alpha_n(L),$$

and that $\omega_i(L)$ is the norm of the map $\wedge^i(L) : \wedge^i E \to \wedge^i E'$ obtained from $L$. It follows that if $L' : E' \to E''$ is another linear map between $n$-dimensional Euclidean spaces, then

$$\omega_i(L'L) \leq \omega_i(L') \cdot \omega_i(L) \text{ for } i = 1, \cdots, n.$$

Consider $T_x f : E_x^s \to E_{f(x)}^s$ and $T_x f^{-1} : E_x^u \to E_{f^{-1}(x)}^u$. As in Temam (1988), we are going to define the "stable" and "unstable" uniform Lyapunov exponents of $f$ on $K$, which can be understood as the uniform Lyapunov exponents on the stable bundle $E^s$, and the unstable bundle $E^u$, respectively.

Let $K$ be a compact hyperbolic set for the $C^1$ diffeomorphism $f$ with $T_K X = E^s \oplus E^u$ as an $Tf$ invariant splitting. Denote that $n_1 = \dim E^s$, and $n_2 = \dim E^u$. Thus the dimension of $X$ is $n_1 + n_2 = n$. The uniform stable Lyapunov exponents and Lyapunov exponents are defined as follows.

**Definition 3.3.1.** *For $1 \leq i \leq n_1$ define $\bar{\omega}_i^s(f) = \sup_{x\in K} \omega_i(T_x f|E_x^s)$. The definition of $\omega_i$ implies that*

$$\bar{\omega}_i^s(f^{p+q}) \leq \bar{\omega}_i^s(f^p) \cdot \bar{\omega}_i^s(f^q).$$

*That gives the subadditivity of $\log \bar{\omega}_i^s(f^k)$. By the subadditivity the limit*

$$\lim_{k\to\infty} \frac{1}{k} \log \bar{\omega}_i^s(f^k)$$

*exists, and is denoted by $\nu_i^s$. Let $\mu_1^s = \nu_1^s$, and $\mu_i^s = \nu_i^s - \nu_{i-1}^s$ for $1 < i \leq n_1$. We call those $\mu_i^s$ the uniform stable Lyapunov exponents of $f$.*

A similar argument to above leads to the uniform unstable Lyapunov exponents, which are defined as follows.

**Definition 3.3.2.** *For $1 \leq i \leq n_2$ define $\bar{\omega}_i^u(f) = \sup_{x\in K} \omega_i(T_x f^{-1}|E_x^u)$. The definition of $\omega_i$ implies that*

$$\bar{\omega}_i^u(f^{p+q}) \leq \bar{\omega}_i^u(f^p) \cdot \bar{\omega}_i^u(f^q).$$

*That gives the subadditivity of $\log \bar{\omega}_i^u(f^k)$. By the subadditivity the limit*

$$\lim_{k\to\infty} \frac{1}{k} \log \bar{\omega}_i^u(f^k)$$

*exists, and is denoted by $\nu_i^u$. Let $\mu_1^u = \nu_1^u$, and $\mu_i^u = \nu_i^u - \nu_{i-1}^u$ for $1 < i \leq n_2$. We call those $\mu_i^u$ the uniform unstable Lyapunov exponents of $f$.*

**Remark:** It is worth to notice that the unstable uniform Lyapunov exponents of $f$ are equal to the stable uniform Lyapunov exponents of $f^{-1}$.

As a matter of fact, the uniform Lyapunov exponents $\mu_i^s$ and $\mu_i^u$ are negative numbers.

## §3.4. The Distortion of a Ball under a Diffeomorphism

At each $x \in X$, the tangent space $T_xX$ is a linear Euclidean space. As in Fathi (1989), define $\pi_s : E \to E^s$, and $\pi_u : E \to E^u$ to be two projections. Let $\|\cdot\|_{E^s}$, and $\|\cdot\|_{E^u}$ be norms on $E^s$ and $E^u$ respectively derived from the Euclidean structure of the tangent bundle $E$. For a vector $v \in E_x = T_xX$, our max norm $\|v\|$ is defined by

$$\|v\|_E = \max\{\|\pi_s(v)\|_{E^s}, \|\pi_u(v)\|_{E^u}\}.$$

We adopt this max norm throughout this chapter.

Locally around $x$ in $X$ is a neighborhood homeomorphic to an open subset in $R^{n_1+n_2} = R^n$. Let $O_x$ denote the origin of the linear space $E_x = T_xX = E_x^s \oplus E_x^u$. Using the exponential map of some Riemannian metric on $X$, we can find a $C^\infty$ map $\theta : \mathcal{U} \to X$, where $\mathcal{U}$ is an open neighborhood of the zero section in $TX$, such that for each $x$ in $X$, the map $\theta_x = \theta|_{\mathcal{U}\cap T_xX}$ is a diffeomorphism onto some open subset in $X$. Also $\theta_x(O_x) = x$ and the derivative of $\theta_x$ at the origin $O_x$ of $T_xX$ is equal to the identity. Of course, the tangent space of $E_x$ at the origin $O_x$ is identified to $E_x$ itself. Since $K$ is compact, we can fix a real number $\delta > 0$ for all $x$ in $K$ that $\theta_x : B(O_x, \delta) \to X$ is well defined , and

$$\frac{1}{4}\|v\| < \|T\theta_x(v)\| < 4\|v\|$$

when $v \in B(O_x, \delta)$. Choose $\delta'_x > 0$ for each $x$ in $X$, such that

$$f(\theta_x(B(O_x, \delta'_x))) \subset \theta_{f(x)}(B(O_{f(x)}, \delta)).$$

Since $K$ is compact, we can fix a small $\delta' > 0$ meeting the needs of all $x \in K$ as $\delta'_x$. If $\delta' < \delta$, write $\delta$ for $\min\{\delta', \delta\}$; noticing that $\theta_x$ is still well-defined on $B(O_x, \delta)$ for each $x \in K$. Define for $v \in B(O_x, \delta)$ the map

$$\tilde{f}_x : B(O_x, \delta) \to E_{f(x)} \text{ by } \tilde{f}_x(v) = \theta^{-1}_{f(x)} \circ f \circ \theta_x(v).$$

Since the derivative of $\theta_x$ at $O_x$ is equal to the identity, the derivative of $\tilde{f}_x$ at the origin $O_x$ is equal to $T_x f$, thus $T\tilde{f}_x(O_x) = T_x f : E_x \to E_{f(x)}$.

Now assume that $f : U \to X$ is a $C^2$ diffeomorphism. Hence $\tilde{f}_x : B(O_x, \delta) \to E_{f(x)}$ is also $C^2$. Using the compactness of $K$ and Taylor's formula, we can write for $v$ near $O_x$ in $T_x X$,

$$\tilde{f}_x(v) = \tilde{f}_x(O_x) + T\tilde{f}_x(O_x)(v) + g_x(v) = T_x f(v) + g_x(v),$$

where $\|g_x(v)\| \le \Delta \|v\|^2$ for some positive number $\Delta$ not depending on $x$. Again, since $K$ is compact, by making $\delta$ smaller if necessary, and $\Delta$ big enough, then for any $v \in B(O_x, \delta)$,

$$\tilde{f}_x(v) = T_x f(v) + g_x(v),$$

where $\|g_x(v)\| \le \Delta \|v\|^2$, and $\Delta > 0$ is a constant not depending on $x$. Repeat the above argument for $f^{-1}$ to get

$$\tilde{f}_x^{-1}(v) = T_x f^{-1}(v) + \tilde{g}_x(v),$$

where $\|\tilde{g}_x(v)\| \le \Delta \|v\|^2$.

LEMMA 3.4.1. *Let $\delta$ be as above. For any positive number $\varepsilon$, there exists a positive number $\eta_0 < \delta$, such that at each $x \in K$ under the maps $\tilde{f}_x$ and $\tilde{f}_x^{-1}$,*

*when $0 < \eta < \eta_0$, the image of a ball centered at $O_x$ in $E_x$ satisfies*

$$\tilde{f}_x(B(O_x,\eta)) \subset (1+\varepsilon)T_xf(B(O_x,\eta)),$$
$$\tilde{f}_x^{-1}(B(O_x,\eta)) \subset (1+\varepsilon)T_xf^{-1}(B(O_x,\eta)),$$

PROOF: This follows easily from the compactness of $K$ and the fact that the map

$$(T_xf)^{-1}\tilde{f}_x : B(O_x,\delta) \to T_xX$$

has the derivative identity at $O_x$.

In order to prove Theorem I, we notice that $N\mu_1, \ldots, N\mu_n$ are the uniform Lyapunov exponents of $f^N$ and the topological entropy $ent(f^N|_K) = N \cdot ent(f|_K)$. So if we can show that Theorem I is true for $f^N$, then it is also true for $f$. Since $AB^2 < 1$, we can let $N$ be large enough that $A_NB_N^2 < 1$. Without loss of generality simply assume $A_1B_1^2 < 1$. The next lemma tells the distortion of balls under iterated maps. The pinching condition is used.

LEMMA 3.4.2. *Define a subset in $E_x$ by*

$$B_m(O_x,\alpha) = \{v \in E_x | \|\tilde{f}_{f^i(x)} \cdots \tilde{f}_x(v)\| < \alpha,$$
$$\|\tilde{f}^{-1}_{f^{-i}(x)} \cdots \tilde{f}_x^{-1}(v)\| < \alpha, i = 0, \ldots, m-1\}$$

*where $\alpha > 0$ and $m$ is a positive integer. Let $f : U \to X$ be a $C^2$ diffeomorphism with a compact hyperbolic set $K$, and $A_1B_1^2 < 1$. Then for any $\varepsilon > 0$, there exists $\eta > 0$, such that*

$$B_m(O_x,\alpha) \subset (1+\varepsilon)^m\{Tf^m_{f^{-m}(x)}(B(O_{f^{-m}(x)},\alpha)) \cap Tf^{-m}_{f^m(x)}(B(O_{f^m(x)},\alpha))\}$$

*for all* $m = 1, 2, \ldots,$ *and all* $x$ *in* $K$, *when* $0 < \alpha < \eta$.

PROOF: Make $\varepsilon$ smaller if necessary so that

$$A_1 B_1^2 (1+\varepsilon) < 1, \text{ and } B_1(1+\varepsilon) < 1.$$

Let $\eta$ be the same as in Lemma 3.4.1, making it smaller if necessary so that $\eta < \varepsilon A_1^{-1} \Delta^{-1}$. Lemma 3.4.1 gives

$$B_1(O_x, \alpha) \subset (1+\varepsilon) Tf(B(O_{f^{-1}(x)}, \alpha)) \cap (1+\varepsilon) Tf^{-1}(B(O_{f(x)}, \alpha)).$$

So the case $m = 1$ follows. Suppose that our conclusion is true for $m$ at all $x \in K$. We show that it is also true for $m+1$ at all $x \in K$ and complete the induction process.

It is clear that $\tilde{f}_x(B_{m+1}(O_x, \alpha)) \subset B_m(O_{f(x)}, \alpha)$, and so

$$B_{m+1}(O_x, \alpha) \subset (\tilde{f}_x)^{-1}(B_m(O_{f(x)}, \alpha)) = \tilde{f}_{f(x)}^{-1}(B_m(O_{f(x)}, \alpha)).$$

Thus for any $v \in B_{m+1}(O_x, \alpha)$, we can have

$$\tilde{f}_{f(x)}^{-1}(u) = v, \text{ where } u \in B_m(O_{f(x)}, \alpha) \text{ exists.}$$

Applying the induction hypothesis at the point $f(x)$ for $m$, we obtain that

$$u \in (1+\varepsilon)^m \{Tf^m(B(O_{f^{-m+1}(x)}, \alpha)) \cap Tf^{-m}(B(O_{f^{m+1}(x)}, \alpha))\},$$

and in particular we can pick some

$$w_1 \in B(O_{f^{m+1}(x)}, \alpha)$$

with $(1+\varepsilon)^m Tf^{-m}(w_1) = u$. Moreover,

$$\|\pi_s(u)\| \leq (1+\varepsilon)^m B_1^m \alpha,$$

and

$$\|\pi_u(u)\| \leq (1+\varepsilon)^m B_1^m \alpha$$

(because $\|Tf^m|_{E^s}\| \leq B_1^m$ and $\|Tf^{-m}|_{E^u}\| \leq B_1^m$). It follows that

$$\|u\| \leq (1+\varepsilon)^m B_1^m \alpha < \alpha < \delta,$$

and thus

$$\tilde{f}_{f(x)}^{-1}(u) = T_{f(x)} f^{-1}(u) + \tilde{g}_{f(x)}(u)$$

is well-defined, with

$$\|\tilde{g}_{f(x)}(u)\| \leq \Delta \|u\|^2 \leq \Delta \cdot (1+\varepsilon)^{2m} B_1^{2m} \alpha^2 \leq \varepsilon (1+\varepsilon)^m \cdot \alpha A_1^{-m-1}.$$

Since $\max_{x \in K} \|T_x f\| \leq A_1$, we have $\|T_x f^{-1}(v)\| \geq A_1^{-1} \|v\|$ for all $x \in K$ and $v \in E_x$. Thus

$$T_x f^{-1}(B(O_x, \alpha)) \supset A_1^{-1} B(O_{f^{-1}(x)}, \alpha),$$

and

$$T_x f^{-n}(B(O_x, \alpha)) \supset A_1^{-n} B(O_{f^{-n}(x)}, \alpha)$$

follows. Hence

$$\tilde{g}_{f(x)}(u) \in \varepsilon (1+\varepsilon)^m A_1^{-m-1}(B(O_x, \alpha)) \subset \varepsilon (1+\varepsilon)^m T f^{-m-1}(B(O_{f^{m+1}(x)}, \alpha)),$$

which allows us to pick some $w_2 \in B(O_{f^{m+1}(x)}, \alpha)$ satisfying

$$\varepsilon(1+\varepsilon)^m Tf^{-(m+1)}(w_2) = \tilde{g}_{f(x)}(u).$$

Setting $w = (1+\varepsilon)^{-1}(w_1 + \varepsilon w_2) \in B(O_{f^{m+1}(x)}, \alpha)$, then

$$\begin{aligned} v &= Tf^{-1}((1+\varepsilon)^m Tf^{-m}(w_1)) + \varepsilon(1+\varepsilon)^m Tf^{-(m+1)}(w_2) \\ &= (1+\varepsilon)^{m+1} Tf^{-m-1}(w), \end{aligned}$$

showing that

$$v \in (1+\varepsilon)^{m+1} Tf^{-(m+1)}(B(O_{f^{m+1}(x)}, \alpha)).$$

Hence we conclude that at $x \in K$, one has

$$B_{m+1}(O_x, \alpha) \subset (1+\varepsilon)^{m+1} Tf^{-(m+1)}(B(O_{f^{m+1}(x)}, \alpha)).$$

Similarly, applying induction hypothesis at $f^{-1}(x)$ and considering the relation

$$(\tilde{f}_x)^{-1}(B_{m+1}(O_x, \alpha)) \subset B_m(O_{f^{-1}(x)}, \alpha),$$

one obtains another relation

$$B_{m+1}(O_x, \alpha) \subset (1+\varepsilon)^{m+1} Tf^{m+1}(B(O_{f^{-(m+1)}(x)}, \alpha)).$$

So the induction hypothesis holds for $m+1$, finishing the proof of Lemma 3.4.2. ∎

In the proof of our Theorem I, we use the following fact:

**Lemma 3.4.3.** *If $L : E \to F$ is a linear isomorphism between two $n$ dimensional linear Euclidean spaces, and $B = B(O, \alpha)$ is a ball of radius $\alpha$, centered $O$ in $E$, then the image $L(B)$ of $B$ under $L$ is an ellipsoid of axes $(\alpha \cdot \alpha_1(L), ..., \alpha \cdot \alpha_n(L))$ in $F$.*

**Proof:** See Temam (1989).

Lemma 3.4.3 shows that $\pi_s T f^m B(O_x, \alpha)$ and $\pi_u T f^{-m} B(O_x, \alpha)$ are ellipsoids with certain axes in $E^s_{f^m(x)}$ and $E^u_{f^{-m}(x)}$, respectively.

## §3.5. The Proof of Theorem I

First, for any $\varepsilon > 0$ , we can find some integer $N > 0$, such that when $k \geq N$, $A_k B_k^2 < 1$ and

$$\frac{1}{k} \log \bar{\omega}_i^s(f^k) < (\mu_1^s + \varepsilon) + \cdots + (\mu_i^s + \varepsilon)$$

for $i = 1, \cdots, n_1$ , as well as

$$\frac{1}{k} \log \bar{\omega}_i^u(f^k) < (\mu_1^u + \varepsilon) + \cdots + (\mu_i^u + \varepsilon)$$

for $i = 1, \cdots, n_2$.

It suffices to prove the theorem for $f^N$ since the entropy and the uniform Lyapunov exponents of $f^N$ are those of $f$ multiplied by $N$. So without loss of generality we simply assume that

$$\log \bar{\omega}_i^s(f) < (\mu_1^s + \varepsilon) + \cdots + (\mu_i^s + \varepsilon)$$

for $i = 1, \cdots, n_1$ and

$$\log \bar{\omega}_i^u(f) < (\mu_1^u + \varepsilon) + \cdots + (\mu_i^u + \varepsilon)$$

for $i = 1, \cdots, n_2$ , and $A_1 B_1^2 < 1$. Relabel $\{\mu_i^s, \mu_i^u\}$ to $\{\mu_1, \cdots, \mu_n\}$ with

$$0 > \mu_1 \geq \cdots \geq \mu_n.$$

Making $\varepsilon$ smaller if necessary assume that $\mu_i + \varepsilon < 0$ for $i = 1, \cdots, n$. Let $\rho_i = \exp(\mu_i + \varepsilon) < 1$ for $i = 1, 2, \cdots, n$, and let $\rho_i^s = \exp(\mu_i^s + \varepsilon) < 1$ for $i = 1, 2, \cdots, n_1$, as well as $\rho_i^u = \exp(\mu_i^u + \varepsilon) < 1$ for $i = 1, 2, \cdots, n_2$. Make $\varepsilon$ even smaller if necessary so that $(1+\varepsilon)\rho_i < 1$, for $i = 1, \cdots, n$.

Define $\tilde{f}_x$ and $B_m(O_x, \alpha)$ as in Section 3.4. By Lemma 3.4.2, there exists $\eta > 0$, such that when $\alpha \leq \eta$,

$$B_m(O_x, \alpha) \subset (1+\varepsilon)^m \{Tf^m(B(O_{f^{-m}(x)}, \alpha)) \cap Tf^{-m}(B(O_{f^m(x)}, \alpha))\}.$$

Using Lemma 3.4.3, we know that $\pi_s Tf^m(B(O_{f^{-m}(x)}, \alpha))$ is an ellipsoid in $E_x^s$ with axes $\{\alpha\rho_{1,m}(x), \cdots, \alpha\rho_{n_1,m}(x)\}$, where $\rho_{i,m}(x) = \alpha_i(Tf^m \mid_{E_x^s})$ for $i = 1, \cdots, n_1$; and it follows that

$$\rho_{1,m}(x) \cdots \rho_{i,m}(x) = \omega_i(T_x f^m|_{E_x^s}) < (\rho_1^s \cdots \rho_i^s)^m.$$

In the mean time one can check that $\pi_u Tf^m(B(O_{f^{-m}(x)}, \alpha)) \supset \pi_u B(O_x, \alpha)$. Similarly, we know that $\pi_u Tf^{-m}(B(O_{f^{-m}(x)}, \alpha))$ is an ellipsoid in $E_x^u$ of axes $\{\alpha\rho_{n_1+1,m}(x), \cdots, \alpha\rho_{n,m}(x)\}$ where $\rho_{n_1+i,m} = \alpha_i(Tf^{-m} \mid E_x^u)$, with

$$\rho_{n_1+1,m}(x) \cdots \rho^m_{n_1+i,m}(x) \leq (\rho_1^u \cdots \rho_i^u)^m,$$

and $\pi_s Tf^{-m}(B(O_{f^m(x)}, \alpha)) \supset \pi_s B(O_x, \alpha)$. Notice that we require

$$\rho_{1,m}(x) \geq \ldots \geq \rho_{n_1,m}(x) \text{ and } \rho_{n_1+1,m}(x) \geq \ldots \geq \rho_{n,m}(x).$$

Therefore the set $Tf^m(B(O_{f^{-m}(x)},\alpha)) \cap Tf^{-m}(B(O_{f^m(x)},\alpha))$ in $E_x = E_x^s \oplus E_x^u$ is a product of two ellipsoids, one in $E_x^s$ and the other in $E_x^u$, with axes

$$\{\alpha\rho_{1,m}(x),\cdots,\alpha\rho_{n_1,m}(x)\}, \text{ and } \{\alpha\rho_{n_1+1,m}(x),\cdots,\alpha\rho_{n,m}(x)\}$$

respectively. By the following fact, if we renumber those $\rho_{i,m}(x)$, such that $\rho_{1,m}(x) \geq \cdots \geq \rho_{n,m}(x)$, and note that $\{\rho_1,\cdots,\rho_n\}$ is actually a reshuffle of these $\rho_i^s$, $\rho_i^u$ so that $\rho_1 \geq \cdots \geq \rho_n$, we can still get $\rho_{1,m}(x)\cdots\rho_{i,m}(x) \leq (\rho_1\cdots\rho_i)^m$ for $i = 1,\cdots,n$.

FACT. *Suppose that $a_1 \geq \cdots \geq a_{n_1}$ and $a_{n_1+1} \geq \cdots \geq a_{n_1+n_2}$, and that $b_1 \geq \cdots \geq b_{n_1}$ and $b_{n_1+1} \geq \cdots \geq b_{n_1+n_2}$. Assume that $a_1\cdots a_i \leq b_1\cdots b_i$ for $i = 1,\cdots,n_1$ and $a_{n_1+1}\cdots a_{n_1+j} \leq b_{n_1+1}\cdots b_{n_1+j}$ for $j = 1,\cdots,n_2$. If we reshuffle those $a_i$ and $b_j$ to get $a'_1 \geq \cdots \geq a'_{n_1+n_2}$ and $b'_1 \geq \cdots \geq b'_{n_1+n_2}$, we still have $a'_1\cdots a'_i \leq b'_1\cdots b'_i$ for $i = 1,\cdots,n_1+n_2$.*

As a product of two ellipsoids, the set

$$Tf^m(B(O_{f^{-m}(x)},\alpha) \cap Tf^{-m}(B(O_{f^m(x)},\alpha))$$

is covered by no more than

$$J\cdot\rho_{1,m}(x)\cdots\rho_{j-1,m}(x)/(\rho_{j,m}(x))^{j-1}$$

balls of radius $\alpha\rho_{j,m}(x)$, where $J$ is a constant depending on the dimension $n$ and the compact set $K$. Hence by Lemma 3.4.2, $B_m(O_x,\alpha)$ is covered by no more than

$$J\cdot\rho_{1,m}(x)\cdots\rho_{j-1,m}(x)/(\rho_{j,m}(x))^{j-1}$$

balls of radius $(1+\varepsilon)^m \alpha \rho_{j,m}(x)$. Since K is compact, we can find a constant $C > 0$ such that for any $x$ in $K$, a subset of the form $\theta_x(B(v,\beta))$ with $\beta < \delta$ in $X$ can be covered by a ball of radius $C\beta$ of $X$ for the metric on $X$ obtained from the Riemannian metric, where $v \in B(O_x, \delta)$. So $\theta_x(B_m(O_x, \alpha))$ can be covered by no more than $J \cdot \rho_{1,m}(x) \cdots \rho_{j-1,m}(x)/(\rho_{j,m}(x))^{j-1}$ balls of radius

$$C(1+\varepsilon)^m \alpha \rho_{j,m}(x) < C(1+\varepsilon)^m \rho_1^m \alpha < C\alpha.$$

Denote by $N(m,\alpha)$ the minimum number of sets of the form $\theta_x(B_m(O_x,\alpha))$ needed to cover $K$. We let $M$ be a subset of $K$ such that

$$\{\theta_x(B_m(O_x,\alpha)) : x \in M\}$$

covers $K$ and the cardinality of $M$ is $N(m,\alpha)$. Since

$$\theta_x(B_m(O_x,\alpha)) \supset \bigcap_{i=-m}^{m} f^{-i}[B(f^i(x), \alpha/C)],$$

by taking $f^{-m}$, we know

$$N(m,\alpha) \le r_{2m+1}(\alpha/C, K),$$

where $r_{2m+1}(\alpha/C, K)$ is the smallest cardinality of a $(2m+1, \alpha/C)$ spanning set for $f\,|_K$ (see Section 3.2 or Walters (1982) for the definition of a spanning set). When $m$ is large enough, and $\alpha$ is small enough, we have

$$r_{2m+1}(\alpha/C, K) < \exp m(2\lambda + \varepsilon),$$

where $\lambda$ is the topological entropy of $f$ over $K$. Thus

$$N(m,\alpha) < \exp m(2\lambda+\varepsilon).$$

Pick the integer $j$ such that $j-1 \le D < j$. Let $\varepsilon' = \varepsilon(n+3)/(-\mu_j) > 0$ be fixed and let $s = D + \varepsilon'$. Here the number $\varepsilon$ is chosen small enough that $j-1 \le s < j$. If $\{B_i : i \in I\}$ is an open cover for $K$, where $B_i$ is a ball of radius $r_i$, we define

$$|I| = \max_{i\in I} |r_i|.$$

By definition (see chapter 2), the Hausdorff pre-measure $\mathcal{H}^s_\varepsilon$ is given by

$$\inf_{|I|<\varepsilon} \sum_{i\in I} r_i^s$$

and has been proved to be a non-increasing function of $\varepsilon$'s. Therefore,

$$\begin{aligned}
\mathcal{H}^s_{C\alpha}(K) &= \inf_{|I|\le C\alpha} \sum_{i\in I} r_i^s \le \mu(C(1+\varepsilon)^m \alpha \rho_1^m, s)\\
&\le \sum_{x\in M} \frac{J\rho_{1,m}(x)\ldots\rho_{j-1,m}^m(x)}{[\rho_{j,m}(x)]^{j-1}}(\rho_{j,m}(x))^s \cdot [C(1+\varepsilon)^m\alpha]^s\\
&\le N(m,\alpha)\cdot \rho_1^m \ldots \rho_{j-1}^m \rho_j^{(s-j+1)m} \cdot [C(1+\varepsilon)^m\alpha]^s \cdot J\\
&\le \exp[m(2\lambda+\varepsilon)] \cdot \exp m(\mu_1 + \ldots + \mu_{j-1} + (s-j+1)\mu_j + s\varepsilon)\\
&\quad \cdot \exp[m\log(1+\varepsilon)] \cdot C^s J \alpha^s\\
&= \exp m[2\lambda + \mu_1 + \ldots + \mu_{j-1} + (s-j+1)\mu_j + (1+s)\varepsilon + \log(1+\varepsilon)] \cdot C^s J\alpha^s\\
&= \exp m[(s+1)\varepsilon + 2\lambda + \tau(D) + \varepsilon'\mu_j + \log(1+\varepsilon)] \cdot C^s J\alpha^s\\
&\le \exp m[(s+1)\varepsilon + \varepsilon'\mu_j + \varepsilon] \cdot C^s J\alpha^s\\
&\le \exp m[(n+2)\varepsilon + (-n-3)\varepsilon] \cdot C^s J\alpha^s\\
&= \exp(-m\varepsilon)\cdot C^s J\alpha^s\\
&\to 0,
\end{aligned}$$

as $m \to \infty$. Thus $\mathcal{H}^s_{C\alpha}(K) = 0$, if $\alpha$ is small enough. So

$$\mathcal{H}^s(K) = \lim_{\alpha \to 0} \mathcal{H}^s_{C\alpha}(K) = 0.$$

Therefore

$$HD(K) \le s = D + \varepsilon(n+3)/(-\mu_j).$$

Let $\varepsilon \to 0$, we get $HD(K) \le D$ as desired. The proof of Theorem I is thus completed. ∎

## §3.6. The Case of a $C^2$ Flow

Let $\{f^t : t \in R\}$ be a $C^2$ partial flow on the Riemannian manifold $X$, and let $K$ be a compact invariant hyperbolic subset of $f^t$. See Fathi (1989) for the definitions. We have results similar to Theorem I about flows. Let $n_1$ and $n_2$ be the dimensions of the stable and unstable foliations. Let us define stable and unstable uniform Lyapunov exponents for a $C^2$ partial flow $f^t$. The subadditivity is used for the existence of limits.

DEFINITION 3.6.1. *For $1 \le i \le n_1$ and $1 \le j \le n_2$, define*

$$\bar{\omega}^s_i(f^t) = \sup_{x \in K} \omega_i(T_x f^t | E^s_x), \bar{\omega}^u_j(f^t) = \sup_{x \in K} \omega_j(T_x f^{-t} | E^u_x).$$

*It follows that for all $p > 0$ and $q > 0$,*

$$\bar{\omega}^s_i(f^{p+q}) \le \bar{\omega}^s_i(f^p) \cdot \bar{\omega}^s_i(f^q)$$

*and*

$$\bar{\omega}^u_j(f^{p+q}) \le \bar{\omega}^u_j(f^p) \cdot \bar{\omega}^u_j(f^q).$$

*So by subadditivity both*

$$\lim_{t\to+\infty}\frac{1}{t}\log\bar{\omega}_i^s(f^t)\ \textit{and}\ \lim_{t\to+\infty}\frac{1}{t}\log\bar{\omega}_i^u(f^t)$$

*exist, and are denoted by* $\nu_i^s$, $\nu_i^u$. *Let* $\mu_1^s=\nu_1^s$ $(\mu_1^u=\nu_1^u)$, *and* $\mu_i^s=\nu_i^s-\nu_{i-1}^s$ *for* $1<i\le n_1$ *(respectively,* $\mu_j^u=\nu_j^u-\nu_{j-1}^u$ *for* $1<j\le n_2$*). We call* $\mu_i^s$ *and* $\mu_j^u$ *the uniform stable, or unstable Lyapunov exponents for* $f^t$.

THEOREM 3.6.2. *Let* $f^t$ *be a* $C^2$ *partial flow with a compact hyperbolic set* $K$ *in the n-dimensional Riemannian manifold* $X$. *Let* $\mu_1^s,\mu_2^s,...,\mu_{n_1}^s$ *be the stable uniform Lyapunov exponents of* $f^t$*; and let* $\mu_1^u,\mu_2^u,...,\mu_{n_2}^u$ *be the unstable uniform Lyapunov exponents of* $f^t$. *Relabel the set* $\{\mu_1^s,\cdots,\mu_{n_1}^s;\mu_1^u,\cdots,\mu_{n_2}^u\}$ *as* $\{\mu_1,\cdots,\mu_n\}$ *with*

$$0>\mu_1\ge\mu_2\ge\cdots\ge\mu_n.$$

*Define* $\tau:[0,n]\to R$ *by*

$$\tau(s)=\mu_1+...+\mu_{[s]}+(s-[s])\mu_{[s]+1}.$$

*Let* $\lambda=ent(f^1|_K)$ *be the topological entropy of* $f^1$ *on* $K$. *Let*

$$D=\tau^{-1}(-2\lambda),$$

*or* $D=n$ *if* $\mu_1+\mu_2+...+\mu_n>-2\lambda$.

*If* $AB^2<1$, *where* $A=\lim A_t$, $B=\lim B_t$, *and*

$$A_t=\max\{\max_{x\in K}\|T_xf^t\|^{\frac{1}{t}},\max_{x\in K}\|T_xf^{-t}\|^{\frac{1}{t}}\},$$
$$B_t=\max\{\max_{x\in K}\|T_xf^t|_{E^s}\|^{\frac{1}{t}},\max_{x\in K}\|T_xf^{-t}|_{E^u}\|^{\frac{1}{t}}\},$$

*then the Hausdorff dimension of the hyperbolic set $K$ of the flow satisfies*

$$HD(K) \leq D + 1.$$

The proof of the above Theorem, which is about the case of follows, is much the same as that of our Theorem I.

REMARK: The geodesic flows of $1/4$ pinched negatively curved Riemannian metrics on compact manifolds satisfy the hypothesis of our Theorem 3.6.2 for any subset of the unit tangent bundle which is invariant under the geodesic flow.

# CHAPTER 4

# THE HYPERBOLIC SET AND TOPOLOGICAL PRESSURE

## §4.1. The Hyperbolic Set Revisited

Now let us use the topological pressures on a hyperbolic set to bound the Hausdorff dimensions of the hyperbolic set and its transverse with the stable and unstable manifolds. For convenience in this chapter, we use an alternative definition of the hyperbolic set. Mather has pointed out that by choosing an appropriate Riemann metric, any set that is hyperbolic according to the previous chapter will satisfy the definition we give here. We prove Theorem II given in the introduction here by proving Theorems 4.2 through 4.4. Our Theorem 4.1 in this chapter gives a generally rough estimate, and we will see the introduction of the pinching condition enables a better estimate.

Let $U$ be an open set in a smooth Riemannian manifold $X$ and $f : U \to X$ a diffeomorphism from $U$ onto an open subset of $X$. If $x \in X$, we denote by $T_xX$ the tangent space of $X$ at $x$, and by $T_xf : T_xX \to T_{f(x)}X$ the derivative of $f$ at $x$. More generally we denote by $T_KX$ the restriction of the tangent bundle of $X$ to a subset $K$ of $X$.

DEFINITION 4.1.1. *Let $K \subset U \subset X$ be a compact subset of a Riemannian manifold $X$. We say that $K \subset U$ is hyperbolic for a diffeomorphism $f$ if $f(K) = K$ and if there is a $Tf$ invariant splitting of the tangent bundle $E = T_KX =$*

*$E^s \oplus E^u$ such that*

$$\max_{x\in K}\{\|T_xf\mid_{E^s}\|, \|T_xf^{-1}\mid_{E^u}\|\} < 1,$$

*where the norm is obtained from an appropriate Riemannian metric on $X$.*

We call $\tau = \max_{x\in K}\{\|T_xf\mid_{E^s}\|, \|T_xf^{-1}\mid_{E^u}\|\}$ the skewness of $K$. This definition of hyperbolicity is slightly different from the ones given in the previous chapter and Fathi (1989). However, if $K$ is hyperbolic in the sense of Chapter 3, we can adapt a metric to $K$ as Mather (1968) did. Then under this Mather metric, $K$ is hyperbolic under our definition here. Refer to Walters (1982) or the next section of this chapter for the definition of the topological pressure for a function $\lambda$ and the map $f|_K : K \to K$, and the variational principle. In this chapter, we let the real valued negative function $\lambda : U \to R$ be defined by

$$\lambda(x) = \log\max\{\|T_xf|_{E^s_x}\|, \|T_xf^{-1}|_{E^u_x}\|\},$$

Clearly the function $\lambda$ is continuous if $f$ is $C^1$. For the Hausdorff dimension of a hyperbolic set $K$, first we have

THEOREM 4.1. *If $f$ is a $C^1$ diffeomorphism with a hyperbolic set $K$, and at some real positive value $s$ we have the topological pressure $P(f|_K, s\lambda) = 0$, then $HD(K) \leq s$.*

We say $f$ satisfies a pinching condition if $AB^2 < 1$ where

$$A = \max_{x\in K}\{\|T_xf\|, \|T_xf^{-1}\|\|\}$$

and $B = \max_{x\in K}\{\lambda(x)\}$. We also define two characteristic functions $\lambda^s$ and $\lambda^u$ as follows.

For $L : E \to F$ a linear map between $n$ dimensional normed spaces $E$ and $F$, we write

$$\alpha_i(L) = \sup_{S \subset E, \dim S = i} \inf_{\|\varphi\|_E = 1, \varphi \in S} \|L\varphi\|_F,$$

and $\omega_i(L) = \alpha_1(L) \cdots \alpha_i(L)$ where $i = 1, \cdots, n$. Here $L$ induces $\wedge^i L : \wedge^i E \to \wedge^i F$; and in fact $\omega_i(L) = \| \wedge^i L\|_F$. If $t$ is not an integer, we write

$$\omega_t(L) = \alpha_1(L) \cdots \alpha_{[t]}(L) \alpha_{[t]+1}(L)^{t-[t]}.$$

It is noticed the linear map $L$ sends any ball of radius $r$ in $E$ to an ellipsoid in $F$ with axes given by $(r\alpha_1(L), \cdots, r\alpha_n(L))$ .

Now consider the splitting of $T_K X = E^s \oplus E^u$. Suppose that $\dim E^s = n_1$, $E^u = n_2$, where $n_1 + n_2 = n$. When $x \in K$ define

$$\lambda^s(t,x) = \log \omega_t(T_x f | E^s_x) \text{ for } t \in [0, n_1],$$

and

$$\lambda^u(t,x) = \log \omega_t(T_x f^{-1} | E^u_x) \text{ for } t \in [0, n_2].$$

Write $\lambda^u_t(x) = \lambda^u(t,x)$ and $\lambda^s_t(x) = \lambda^s(t,x)$. Under the pinching condition and using functions $\lambda^s$ and $\lambda^u$, we have

THEOREM 4.2. *If $f$ is a $C^2$, pinched diffeomorphism with a hyperbolic set $K$, then there is a unique real non-negative value $t$ such that*

$$\max\{P(f|_K, \lambda^s_{t_1}) + P(f|_K, \lambda^u_{t_2}) | t_1 + t_2 = t\} = 0.$$

*Moreover the Hausdorff dimension of the hyperbolic set $K$ satisfies $HD(K) \leq t$.*

We also give the upper bounds for the Hausdorff dimension of the transversals of the hyperbolic set to the stable and unstable manifolds.

THEOREM 4.3. *If $f$ is a $C^2$, pinched diffeomorphism with a hyperbolic set $K$, and at some real positive value $t$ we have the topological pressure $P(f|_K, \lambda_t^s) = 0$, then the Hausdorff dimension in the direction that transverses to $W^u(x, f)$ is given by*

$$HD(K \cap W^s(x, f)) \leq t.$$

THEOREM 4.4. *If $f$ is a $C^2$, pinched diffeomorphism with a hyperbolic set $K$, and at some real positive value $t$ we have the topological pressure $P(f|_K, \lambda_t^u) = 0$, then the Hausdorff dimension in the direction that transverses to $W^s(x, f)$ is given by*

$$HD(K \cap W^u(x, f)) \leq t.$$

REMARK: The inequalities in Theorems 4.3 and 4.4 are equalities when $K$ is the basic set of a two dimensional horseshoe. This fact can be found in McCluskey and Manning (1983).

## §4.2. Topological Pressure

We first give the definition of topological pressure using the open coverings. Then we also give the definition using either spanning sets or separated sets. Both definitions are equivalent. See Walters (1982) for more details and the variational principle. The concept of topological pressure in this type of setting was introduced by Ruelle and studied in the general case by Walters.

Let $f : X \to X$ be a continuous transformation of a compact metric space $(X, d)$. Let $C(X, R)$ denote the Banach algebra of real-valued continuous functions on $X$ equipped with the supremum norm. The topological pressure of $f$ will be a map $P(f, .) : C(X, R) \to R \cup \{\infty\}$ which will have good properties relative

to the structures on $C(X,R)$. It contains topological entropy in the sense that $P(f,0) = ent(f)$ where 0 denote the member of $C(X,R)$ which is identically zero.

Let $\lambda \in C(X,R)$. For $x \in X$ denote $S_n\lambda(x) = \sum_{i=0}^{n-1} \lambda(f^i x)$. Let $\alpha$ be an open cover of $X$ and denote

$$q_n(f,\lambda,\alpha) = \inf\{\sum_{B\in\beta} \inf_{x\in B} \exp((S_n\lambda)(x)) : \beta \text{ is a finite subcover of } \vee_{i=0}^{n-1} f^{-i}\alpha\},$$
$$p_n(f,\lambda,\alpha) = \inf\{\sum_{B\in\beta} \sup_{x\in B} \exp((S_n\lambda)(x)) : \beta \text{ is a finite subcover of } \vee_{i=0}^{n-1} f^{-i}\alpha\}.$$

Notice that $q_n(f,\lambda,\alpha) \leq p_n(f,\lambda,\alpha)$ and the subadditivity of $\log p_n(f,\lambda,\alpha)$ is due to the relation $p_n(f,\lambda,\alpha)\cdot p_m(f,\lambda,\alpha) \geq p_{n+m}(f,\lambda,\alpha)$. Let

$$q(f,\lambda,\alpha) = \limsup_{n\to\infty} \frac{1}{n}\log q_n(f,\lambda,\alpha),$$
$$p(f,\lambda,\alpha) = \lim_{n\to\infty} \frac{1}{n}\log p_n(f,\lambda,\alpha).$$

Recall $|\alpha|$ denotes the largest diameter of sets in $\alpha$. Write

$$q(f,\lambda,\varepsilon) = \sup_{\alpha}\{q(f,\lambda,\alpha) : \alpha \text{ is an open cover of } X, |\alpha| \leq \varepsilon\},$$
$$p(f,\lambda,\varepsilon) = \sup_{\alpha}\{p(f,\lambda,\alpha) : \alpha \text{ is an open cover of } X, |\alpha| \leq \varepsilon\}.$$

Now both sides of the following identity gives the quantity $P(f,\lambda)$ which is called the topological pressure:

$$\lim_{\varepsilon\to 0} q(f,\lambda,\varepsilon) = \lim_{\varepsilon\to 0} p(f,\lambda,\varepsilon).$$

We have defined the $(n,\varepsilon)$ spanning set and the $(n,\varepsilon)$ separated set in the previous chapter. Now we can use them to give alternative definitions of the

topological pressure. Let

$$Q_n(f,\lambda,\varepsilon) = \inf\{\sum_{x\in F} \exp((S_n\lambda)(x)) : F \text{ is } (n,\varepsilon) \text{ spanning for } X\},$$
$$P_n(f,\lambda,\varepsilon) = \sup\{\sum_{x\in F} \exp((S_n\lambda)(x)) : F \text{ is a } (n,\varepsilon) \text{ separated subset of } X\}.$$

Then define

$$Q(f,\lambda,\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log Q_n(f,\lambda,\varepsilon),$$
$$P(f,\lambda,\varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log P_n(f,\lambda,\varepsilon),$$

Now the topological pressure is given by

$$P(f,\lambda) = \lim_{\varepsilon\to 0} Q(f,\lambda,\varepsilon) = \lim_{\varepsilon\to 0} P(f,\lambda,\varepsilon).$$

The variational principle shows the topological pressure can be computed using $f$ invariant measures.

THEOREM 4.2.1. *(The Variational Principle) Let $f : X \to X$ be a continuous self mapping of a compact metric space and $\lambda$ a real valued function defined on $X$. Then the topological pressure can be given by*

$$P(f,\lambda) = \sup\{h_\mu(f) + \int \lambda d\mu : \mu \in M(X,f)\}$$

*where $h_\mu(f)$ is the entropy of $f$ as a map that preserves the measure $\mu$ and $M(X,f)$ is the set of all measures invariant under $f$.*

See Walters (1982) for an introduction to the properties of the topological pressure, as well as the proof of the variational principle and equilibrium states, and how pressure determines the invariant measures.

## §4.3. A Rough Estimate on Hausdorff Dimension

Lemmas 4.3.1 and 4.3.2 are obvious.

LEMMA 4.3.1. *If $f : K \to K$ with $K$ compact and $\lambda_1 < \lambda_2$ are two real functions on $K$, then $P(f, \lambda_1) < P(f, \lambda_2)$.*

LEMMA 4.3.2. *If $\lambda$ is a negative real valued function then $P(f|_K, s\lambda)$ is a strictly decreasing, continuous function of $s$.*

For $x \in K$ let $T_xX = E^s_x \oplus E^u_x$ be the tangent space and $\pi^s_x : T_xX \to E^s_x$, $\pi^u_x : T_xX \to E^u_x$ be the projections. If $v \in T_xX$, and if $\|\pi^s_x(v)\|_{E^s_x}$, $\|\pi^u_x(v)\|_{E^u_x}$ are norms induced from the adapted metric, then the norm

$$\|v\| = \max\{\|\pi^s_x(v)\|_{E^s_x}, \|\pi^u_x(v)\|_{E^u_x}\}$$

is equivalent to the one induced from the adapted metric. With the new norm $K$ is still hyperbolic under our definition. Let $\theta_x : B(0_x, \alpha) \to X$ be the exponential map. Since $K$ compact and since $T\theta_x = I$ the identity, we can find $\delta > 0$ such that $\theta_x$ is well-defined for all $x \in K$ and $\alpha < \delta$, with

$$\frac{1}{2}\|v - w\| \leq d(\theta_x(v), \theta_x(w)) \leq 2\|v - w\|.$$

Define $\tilde{f}_x = \theta^{-1}_{f(x)} f \theta_x$ to be the lifting of $f$. Define the set

$$B_m(O_x, \alpha) = \{v \in E_x | \|\tilde{f}_{f^i(x)} \cdots \tilde{f}_x(v)\| < \alpha,$$
$$\|\tilde{f}^{-1}_{f^{-i}(x)} \cdots \tilde{f}^{-1}_x(v)\| < \alpha, i = 0, \ldots, m-1\}$$

as in Chapter 3.

PROPOSITION 4.3.3. *Let $f$, $K$, be as in Theorem 3.1. If $P(f|_K, s\lambda) < 0$, then the Hausdorff dimension $HD(K) \le s$.*

PROOF: Write $\tilde{f}_x(v) = T_x f(v) + g(v)$. Let $\varepsilon$ be a small number such that

$$\varepsilon < -P(f, s\lambda) \cdot \frac{1}{3}s.$$

There is $\delta > 0$ with $\tilde{f}_x B(0_x, \alpha) \subset (1+\varepsilon)T_x f B(0_x, \alpha)$ when $\alpha < \delta$. Thus $B_m(0_x, \alpha) \subset (1+\varepsilon)^m B(0_x, r_m(x))$ where

$$r_m(x) = \alpha \cdot \max\{\exp(\tilde{S}_m\lambda(x)), \exp(S_m\lambda(x))\},$$

with $\tilde{S}_m\lambda(x) = \sum_{i=1}^m \lambda(f^{-i}x)$, and $S_m\lambda(x) = \sum_{i=1}^m \lambda(f^i x)$. Notice that

$$r_m(x) \le \alpha[\exp \tilde{S}_m\lambda(x) + \exp S_m\lambda(x)].$$

Let $K_1 \subset K$ be a maximal $(2m+1, \frac{1}{4}\alpha)$ separated subset. Then for any $x \in K \cap K_1^c$, there exists $y \in K_1$ such that

$$\max\{d(x,y), \cdots, d(f^{2m+1}(x), f^{2m+1}(y))\} \le \frac{1}{4}\alpha.$$

So $\{\theta_x \tilde{f}^{-m} B_m(0_{f^m(x)}, 2\alpha) | x \in K_1\}$ is an open cover for $K$. Since $\theta\tilde{f} = f\theta$, it follows that $f^{-m}\theta_{f^m(x)} B_m(0_{f^m(x)}, 2\alpha) | x \in K_1\}$ is an open cover for $K$. Let $K_2 = f^m K_1$ which is clearly $(m, \frac{1}{4}\alpha)$ separated. $\{\theta_x B_m(0_x, 2\alpha) | x \in K_2\}$ is also an open cover for $K$. Let the number $M$ be large enough and $\delta$ small enough that

$$\frac{1}{m} \cdot \log P_m(f, s\lambda, \frac{1}{4}\alpha) < P(f, s\lambda, \frac{1}{4}\alpha) + s\varepsilon/2 \le P(f, s\lambda) + s\varepsilon < -2s\varepsilon.$$

Here as in section 4.2,

$$P_m(f, s\lambda, \frac{1}{4}\alpha) = \sup\{\sum_{x\in E} \exp S_m(s\lambda)(x)|E \text{ is an } (m, \frac{1}{4}\alpha) \text{ separated subset of } K\},$$

and

$$P(f, s\lambda, \frac{1}{4}\alpha) = \limsup_{m\to\infty} \frac{1}{m} \log P_m(f, s\lambda, \frac{1}{4}\alpha)$$
$$P(f, s\lambda) = \lim_{\alpha\to 0} P(f, s\lambda, \frac{1}{4}\alpha).$$

Since $P(f^{-1}, s\lambda) = P(f, s\lambda) < 0$, we can require

$$\frac{1}{m} \log P_m(f^{-1}, s\lambda, \frac{1}{4}\alpha) < -2s\varepsilon.$$

Thus we have

$$P_m(f, s\lambda, \frac{1}{4}\alpha) < \exp(-2ms\varepsilon) \text{ and } P_m(f^{-1}, s\lambda, \frac{1}{4}\alpha) < \exp(-2ms\varepsilon).$$

In particular, since $P_m(f, s\lambda, \frac{1}{4}\alpha)$ is the supremum,

$$\sum_{x\in K_2} \exp(S_m s\lambda(x)) < \exp(-2ms\varepsilon), \text{ and } \sum_{x\in K_2} \exp(\tilde{S}_m s\lambda(x)) < \exp(-2ms\varepsilon).$$

Now consider the Hausdorff $s$ measure corresponding to the open cover

$$\{\theta_x B_m(0_x, 2\alpha)|x \in K_2\}.$$

The radius of each open set $\theta_x B_m(0_x, 2\alpha)$ is less than $4\alpha$. Let $\alpha < \delta/2$. Thus

$$\begin{aligned}
\mathcal{H}^s_{4\alpha}(K) &\leq \sum_{x \in K_2} |\theta_x B_m(0_x, 2\alpha)|^s \\
&\leq 4^s \sum_{x \in K_2} |B_m(0_x, 2\alpha)|^s \\
&\leq 4^s \sum_{x \in K_2} (2\alpha)^s (1+\varepsilon)^{sm} (\exp S_m\lambda(x) + \exp \tilde{S}_m\lambda(x)) \\
&< (8\alpha)^s \exp(sm\varepsilon) 2 \exp(-2ms\varepsilon) \\
&= 2(8\alpha)^s \exp(-sm\varepsilon) \to 0,
\end{aligned}$$

as $m \to \infty$. So $\mathcal{H}^s_{4\alpha}(K) = 0$; and $HD(K) \leq s$. ∎

Lemma 4.3.1 and Proposition 4.3.3 give a proof of Theorem 4.1.

PROOF OF THEOREM 4.1: Let $t = \inf\{s |\ P(f|_K, s\lambda) < 0\}$. Then $HD(K) \leq t$. It is clear that $P(f|_K, t\lambda) = 0$ and $t$ is unique in view of Lemma 4.3.1. ∎

## §4.4. Under the Pinching Condition

The following results will prove Theorem 4.2.

PROPOSITION 4.4.1. *Let $f$ be $C^2$, pinched, and $K$ its hyperbolic set. If $t$ is such that*

$$\max_{t_1+t_2=t} \{P(f|_K, \lambda^s_{t_1}) + P(f|_K, \lambda^u_{t_2})\} < 0,$$

*then the Hausdorff dimension $HD(K) \leq t$.*

PROOF: By Lemma 3.4.2, for any $\varepsilon > 0$ there is $\delta > 0$, when $r < \delta$,

$$B_m(0_x, r) \subset (1+\varepsilon)^m \{T_{f^{-m}(x)} f^m B(0_{f^{-m}(x)}, r) \cap T_{f^m(x)} f^{-m} B(0_{f^m(x)}, r)\}$$

Thus, considering their projections we have

$$\pi_s B_m(0_x, r) \subset (1+\varepsilon)^m T_{f_{-m}(x)} f^m \pi_s B(0_{f^{-m}(x)}, r),$$

and

$$\pi_u B_m(0_x, r) \subset (1+\varepsilon)^m T_{f_m(x)} f^{-m} \pi_u B(0_{f^m(x)}, r).$$

Reshuffle

$$\alpha_1(T_{f^{-m}(x)} f^m|_{E^s}), \cdots, \alpha_{n_1}(T_{f^{-m}(x)} f^m|_{E^s});$$
$$\alpha_1(T_{f^m(x)} f^{-m}|_{E^u}), \cdots, \alpha_{n_2}(T_{f^m(x)} f^{-m}|_{E^u})$$

to get $\alpha_1(m,x) \le \cdots \le \alpha_n(m,x)$. It follows that $B_m(0_x, r)$ is contained in an "ellipsoid" of axes $\{r\alpha_1(m,x), \cdots, r\alpha_n(m,x)\}$.

Let $j = [t]+1$. Use balls of radius $r\alpha_j(m,x)$ to cover the above ellipsoid of axes $\{r\alpha_1(m,x), \cdots, r\alpha_n(m,x)\}$. At most

$$Q\alpha_1(m,x)\cdots\alpha_j(m,x)/\alpha_j(m,x)^j$$

balls are needed where $Q$ depends only on the dimension $n$, the metric we choose and the compact hyperbolic set $K$. Notice that the number $\alpha_j(m,x) < 1$.

Let $K_1$, $K_2$ be two subsets of $K$ as in the proof of Proposition 4.3.3, where the collection $\{\theta_x B_m(0_x, r) | x \in K_2\}$ is an open cover for $K$. The balls of radius $r\alpha_j(m,x)$ needed to cover $B_m(0_x, r)$ is at most

$$Q\alpha_1(m,x)\cdots\alpha_j(m,x)/\alpha_j(m,x)^j.$$

The image in $K$ under $\{\theta_x : x \in K_2\}$ is an open cover for $K$, with radius less

than $4r$, and the corresponding Hausdorff $t$ measure is estimated as the following:

$$\begin{aligned}
\mathcal{H}^t_{4r}(K) &\le (1+\varepsilon)^m \sum_{x\in K_2} r^t Q 4^t \cdot \alpha_1(m,x)\cdots\alpha_j(m,x)/\alpha_j(m,x)^j \cdot \alpha_j(m,x)^t \\
&= C(1+\varepsilon)^m \sum_{x\in K_2} \alpha_1(m,x)\cdots\alpha_{j-1}(m,x)\alpha_j(m,x)^{t-j+1} \\
&= C(1+\varepsilon)^m \sum_{x\in K_2} \omega_{t_2(x)}(T_{f^m(x)}f^{-m}|_{E^u})\omega_{t_1(x)}(T_{f^{-m}(x)}f^m|_{E^s}) \\
&< C\exp(m\varepsilon) \sum_{x\in K_2} \exp\sum_{i=1}^{m}[\lambda_{s,t_1(x)}(f^{-i}x) + \lambda_{u,t_2(x)}(f^i x)],
\end{aligned}$$

Here $C = r^t Q 4^t$. At least one of $t_1, t_2$ is an integer. Also $t_1 + t_2 = t$. There are at most $n(n-1)$ pairs of such $(t_1(x), t_2(x))$ for each $x$. For each pair of available $(t_1, t_2)$, we have

$$P(f^{-1}|_K, \lambda^s_{t_1}) + P(f|_K, \lambda^u_{t_2}) < 0.$$

Thus we can require

$$\frac{1}{m}\log[P_m(f^{-1}|_K, \lambda_{s,t_1})P_m(f|_K, \lambda_{u,t_2})] < -2\varepsilon$$

for all available pairs $(t_1, t_2)$ and all large $m$. So,

$$\begin{aligned}
\mathcal{H}^t_{4r}(K) &\le C\exp(m\varepsilon) \sum_{t_1+t_2=t} \{\sum_{x\in K_2} \exp(\sum_{i=1}^{m}\lambda_{u,t_2}(f^i x)) \cdot \sum_{x\in K_2}\exp(\sum_{i=1}^{m}\lambda_{s,t_1}(f^{-i}x))\} \\
&\le C\exp(m\varepsilon) \sum_{t_1+t_2=t} P_m(f^{-1}|_K, \lambda_{s,t_1})P_m(f|_K, \lambda_{u,t_2}) \\
&\le C\exp(m\varepsilon)\exp(-m\varepsilon) \\
&= C\exp(-m\varepsilon) \to 0.
\end{aligned}$$

Therefore $\mathcal{H}^t_{4r}(K) = 0$; and $HD(K) \le t$. ∎

The proof of Theorem 4.2 follows immediately from the above Proposition and Lemma 4.3.2.

## §4.5. The Transversals

By the Stable Manifold Theorem, the stable manifold $W^s(x,f)$ is locally embedded in $X$, with $T_xW^s(x,f) = E^s_x$. Similarly $T_xW^u(x,f) = E^u_x$. Now let

$$\theta^s_x : T_xW^s(x,f) = E^s_x \to W^s(x,f) \subset X$$

be the exponential map for $W^s(x,f)$ at $x$ and

$$\theta^u_x : T_xW^u(x,f) = E^u_x \to W^u(x,f) \subset X$$

be the exponential map for $W^u(x,f)$ at $x$. Define $\zeta_x : T_xX = E^s_x \oplus E^u_x \to T_xX$ by

$$\zeta_x(w \oplus v) = \theta_x^{-1}\theta^s_x(w) + \theta_x^{-1}\theta^u_x(v)$$

for $w \in E^s_x$ and $v \in E^u_x$. Now the exponential map $\theta_x$ can be modified to obtain a map

$$\psi_x : T_xX = E^s \oplus E^u \to X$$

by $\theta_x\zeta_x = \psi_x$, with the property that $T\psi_x(0_x) =$ identity. By the stable manifold theorem, $T_xW^s(x,f) = E^s_x$ and

$$\psi_x(E^s_x \cap B(0_x,r)) \subset W^s(x,f).$$

It is noticed that $\psi_x$ is defined locally and since $K$ compact there is $r > 0$ such that $\psi_x$ is well defined on $B(0_x,r) \subset T_xX$ for all $x \in K$. Then similarly to Chapter 3, we let $\tilde{f}(v) = \psi^{-1}_{f(x)} f\psi_x(v)$ and define the set $B_m(0_x,r)$ through $\tilde{f}$. A distortion lemma similar to Lemma 3.4.2 can be obtained.

PROPOSITION 4.5.1. *Let $f$ and $K$ be as in Theorem 4.3. For any $x \in K$, let $W^s(x,f)$ be the stable manifold at $x$. If the topological pressure $P(f, \lambda_t^s) < 0$, then the Hausdorff dimension $HD(K \cap W^s(x,f)) \leq t$.*

PROOF: Similar to Lemma 3.4.2,

$$B_m(0_x, r) \subset (1+\varepsilon)^m \{T_{f^{-m}(x)} f^m B(0_{f^{-m}(x)}, r) \cap T_{f^m(x)} f^{-m} B(0_{f^m(x)}, r)\},$$

we have

$$\pi_s B_m(0_x, r) \subset (1+\varepsilon)^m T_{f_{-m}(x)} f^m \pi_s B(0_{f^{-m}(x)}, r).$$

The right side is an ellipsoid of axes $(1+\varepsilon)^m r\{\alpha_1(m,x), \cdots, \alpha_{n_1}(m,x)\}$. Let $j = [t]+1$. The number of balls of radius $r\alpha_j(m,x)$ needed to cover $\pi_s B_m(0_x, r)$ is at most

$$C\alpha_1(m,x)\cdots\alpha_j(m,x)/\alpha_j(m,x)^j.$$

Let $M_1 \subset K \cap W^s(f^{-m}x, f)$ be a maximal $(2m+1, \frac{1}{4}r)$ separated subset, such that $M_1 \cup \{y\}$ is not $(2m+1, \frac{1}{4}r)$ separated if $y \in K \cap W^s(f^{-m}x, f)$ and $y \notin M_1$. Then the set $M_2$ given by

$$M_2 = f^m M_1 \subset K \cap W^s(x, f)$$

is maximal $(m, \frac{1}{4}r)$ separated; and $\{\psi_x \pi_s B_m(0_x, r) | x \in M_2\}$ is an open cover for $K \cap W^s(x,f)$. Under the collection of maps $\{\psi_x | x \in M_2\}$, the image of small balls of radius $r\alpha_j(m,x)$ used to cover $B_m(0_x, f)$ have radius less than $4r$. The corresponding Hausdorff $t$ measure for $K \cap W^s(x,f)$, the quantity $\mathcal{H}^t_{4r}(K \cap W^s(x,f))$ is less than

$$\sum_{x \in M_2} 4Cr(1+\varepsilon)^m \alpha_j(m,x)^t \alpha_1(m,x)\cdots\alpha_j(m,x)/\alpha_j(m,x)^j$$

Since

$$
\begin{aligned}
&\alpha_j(m,x)^t \alpha_1(m,x)\cdots\alpha_j(m,x)/\alpha_j(m,x)^j \\
&= \alpha_1(m,x)\cdots\alpha_{j-1}(m,x)\alpha_j(m,x)^{t-j+1} \\
&= \omega_t(T_{f^{-m}x}f^m|E^s) \\
&\le \omega_t(T_{f^{-m}x}f|E^s)\omega_t(T_{f^{-m+1}x}f|E^s)\cdots\omega_t(T_{f^{-1}x}f|E^s) \\
&= \exp\lambda_{s,t}(f^{-m}x)\cdots\exp\lambda_{s,t}(f^{-1}x) \\
&= \exp\sum_{i=1}^{m}\lambda_{s,t}(f^{-i}x)
\end{aligned}
$$

and $r(1+\varepsilon)^m\alpha_j(m,x) < r$, we have

$$
\begin{aligned}
\mathcal{H}^t_{4r}(K\cap W^s(x,f)) &\le 4Cr(1+\varepsilon)^m\sum_{x\in M_2}\exp(\sum_{i=1}^{m}\lambda_{s,t}f^{-i}x) \\
&< 4Cr\exp(m\varepsilon)P_m(f,\lambda_{s,t},\frac{1}{4}r) \\
&\le 4Cr\exp(m\varepsilon)\exp(-2m\varepsilon)\to 0,
\end{aligned}
$$

as $m\to\infty$. So $\mathcal{H}^t_{4r}(K\cap W^s(x,f)) = 0$. It follows that $\mathcal{H}^t(K\cap W^s(x,f)) = 0$ and the Hausdorff dimension $HD(K\cap W^s(x,f)) \le t$. ∎

The proof of of Theorem 4.3 follows immediately from the above Proposition. The proof of Theorem 4.4 works similarly.

# CHAPTER 5

# THE SELF SIMILAR SETS

## §5.1. The Self Similar Sets

In this chapter we shall construct the self similar sets. Then we use the topological pressure defined in Chapter 4 to bound the Hausdorff dimension of self similar sets. Finally we discuss the continuity of Hausdorff dimension at conformal constructions.

The construction of a self similar set starts with a $k \times k$ matrix $A = (a_{ij})$ which has entries zeroes and ones, with all entries of $A^N$ positive for some $N > 0$, see Bedford (1988). For each non-zero $a_{ij}$ we give a contraction map $\varphi_{ij} : R^l \to R^l$ with $\|\varphi_{ij}(x) - \varphi_{ij}(y)\| \leq c\|x - y\|$, where $c < 1$ is a constant and we are using the Euclidean norm on $R^l$. Define the Hausdorff metric by

$$d(E, F) = \inf\{\delta | d(x, F) \leq \delta \text{ for all } \ x \in E, \text{ and } \ d(y, E) \leq \delta \text{ for all } \ y \in F\}$$

in the space $\mathcal{C}$ of all nonempty compact subsets of $R^l$. See, for example, Hutchinson (1981) or Falconer (1990). The map $\Phi$ on the $k$-fold product space $\mathcal{C}^k$ given by

$$\Phi(F_1, \cdots, F_l) = (\cup_{j=1}^k \varphi_{1j}(F_j), \cdots, \cup_{j=1}^k \varphi_{kj}(F_j))$$

is a contraction map. By the Banach Fixed Point Theorem the contraction map $\Phi$ has a unique fixed point in $\mathcal{C}^k$, i.e., a vector of compact non-empty subsets of

$R^l$, denoted by $(E_1, \ldots, E_k) \in \mathcal{C}^k$, with

$$\cup_{a_{ij}=1} \varphi_{ij}(E_j) = E_i.$$

The union $E = \cup_{i=1}^k E_i$ is called a self similar set.

Let

$$\Sigma = \Sigma_A^+ = \{(x_0, x_1, \ldots, x_n, \ldots) | 1 \leq x_i \leq k \text{ and } \ a_{x_i x_{i+1}} = 1 \text{ for all } \ i \geq 0\}$$

be the shift space with the following metric: for $\underline{x} = (x_0, x_1, \ldots)$, $\underline{y} = (y_0, y_1, \ldots)$ in $\Sigma$, $d(\underline{x}, \underline{y}) = 2^{-n}$ if and only if $n = \min\{m | x_m \neq y_m\}$. Let $\sigma$ be the shift map of $\Sigma$ and let $\pi : \Sigma \to E$ be given by

$$\pi(x_0, x_1, \ldots, x_n, \ldots) = \text{ the only point in } \cap_{n \geq 1} \varphi_{x_0 x_1} \varphi_{x_1 x_2} \cdots \varphi_{x_n x_{n+1}}(E_{x_{n+1}}).$$

It is clear that $\pi$ is a Hölder continuous, surjective map. We will denote the composition $\varphi_{x_0 x_1} \cdots \varphi_{x_{n-1} x_n}$ by $\varphi_{x_0 \cdots x_n}$. Also, we assume all $\varphi_{ij}$ be $C^1$ diffeomorphisms and denote the derivative of $\varphi_{ij}$ at a point $x$ by $T_x \varphi_{ij}$ or $T\varphi_{ij}(x)$.

DEFINITION 5.1.1. *The $j$-th Lyapunov number of a linear map $L$, denoted by $\alpha_j(L)$, is the square root of the $j$-th largest eigenvalue of $LL^*$, where $L^*$ is the conjugate of $L$. Write*

$$\omega_t(L) = \alpha_1(L) \cdots \alpha_{[t]}(L) \alpha_{[t]+1}(L)^{t-[t]}.$$

*For a set of construction diffeomorphisms $\{\varphi_{ij}\}$, the function $\lambda_t : \Sigma \to R$ for each $t \in [0, l]$ and $\underline{x} = (x_0 x_1 \ldots) \in \Sigma$ is defined by*

$$\begin{aligned} \lambda_t(\underline{x}) &= \log \alpha_1(T\varphi_{x_0 x_1}(\pi\sigma\underline{x})) + \cdots + \log \alpha_{[t]}(T\varphi_{x_0 x_1}(\pi\sigma\underline{x})) \\ &\quad + (t - [t]) \log \alpha_{[t]+1}(T\varphi_{x_0 x_1}(\pi\sigma\underline{x})) \\ &= \log \omega_t(T\varphi_{x_0 x_1}(\pi\sigma\underline{x})). \end{aligned}$$

The constructions and dimensions of self similar sets have been studied by several authors under various restrictions. In this chapter we relax the restrictions on construction diffeomorphisms to a $\kappa$ pinching condition, which is defined as follows.

DEFINITION 5.1.2. *We say that a $C^1$ smooth homeomorphism $\varphi_{ij}$ satisfies the $\kappa$-pinching condition if the derivatives satisfy the following*

$$\|T_x\varphi_{ij}\|^{1+\kappa} \cdot \|T_{\varphi_{ij}(x)}\varphi_{ij}^{-1}\| < 1$$

*for all $x \in E$.*

REMARK: If $T_x\varphi_{ij}T_x\varphi_{ij}^*$ has eigenvalues

$$\alpha_{1,ij}(x)^2 \geq \cdots \geq \alpha_{l,ij}(x)^2$$

where $T_x\varphi_{ij}^*$ denotes the conjugate of $T_x\varphi_{ij}$, the numbers $\alpha_{1,ij}(x), \ldots, \alpha_{l,ij}(x)$ are Lyapunov numbers with

$$1 > \alpha_{1,ij}(x) \geq \cdots \geq \alpha_{l,ij}(x) > 0.$$

Then the pinching condition is equivalent to $\alpha_{1,ij}(x)^{1+\kappa} < \alpha_{l,ij}(x)$.

For the definition and properties of Hausdorff dimension, refer to Chapter 2 or Kahane (1985). Also, we use the definitions and notions of Walters (1982) in the discussions concerning topological pressure, see also Chapter 4. Theorem III given in the introduction is proved in this chapter through Theorem 5.1 and Theorem 5.2.

THEOREM 5.1. *Let $\{\varphi_{ij}\}$ be the $C^1$ construction diffeomorphisms for the self similar set $E$, satisfying the $\kappa$ pinching condition for some positive number $\kappa \le 1$. Suppose the derivatives of all $\{\varphi_{ij}\}$ are Hölder continuous of order $\kappa$. If $t$ is the unique positive number such that the topological pressure $P(\sigma, \lambda_t) = 0$, then the Hausdorff dimension $HD(E) \le t$.*

Let us recall the disjoint open set condition on the construction of self similar sets, see Hutchinson (1981). It states that for each integer $i$ from 1 to $k$ there is a non-empty open set $U_i$ such that

$$\bigcup_{a_{ij}=1} \varphi_{ij}(U_j) \subset U_i, \quad \text{and} \quad \varphi_{ij}(U_j) \cap \varphi_{ik}(U_k) = \phi \quad \text{if} \quad j \ne k.$$

For $n \ge 0$, denote $U_n(\underline{x}) = \varphi_{x_0x_1}\varphi_{x_1x_2}\cdots\varphi_{x_{n-1}x_n}(U_{x_n})$. It follows that $E_i \subset \bar{U}_i$; and that the collection $\{U_n(\underline{x}) : \underline{x} \in \Sigma\}$ is pairwisely disjoint for each fixed $n$.

THEOREM 5.2. *Let $\{\varphi_{ij}\}$ be the $C^1$ construction diffeomorphisms for the self similar set $E$ in $R^l$, satisfying both the $\kappa$ pinching condition for some positive number $\kappa \le 1$, and the disjoint open set condition. Suppose the derivatives of all $\{\varphi_{ij}\}$ are Hölder continuous of order $\kappa$. If $t$ is the unique positive number such that the topological pressure $P(\sigma, \lambda_t) = 0$, then the Hausdorff dimension*

$$HD(E) \ge \frac{t}{1+\kappa} - l\kappa.$$

REMARK: We call a $C^1$ diffeomorphism $C^{1+\kappa}$ if its derivative is Hölder continuous of order $\kappa$. If we fix the construction to be $C^{1+\beta}$ for some $\beta > 0$ but let $\kappa \to 0$ for the $\kappa$ pinching condition, then our upper and lower bounds will coincide with the estimate for conformal cases as in Bedford (1988).

Theorem 5.1 is proved in Section 5.2, and Theorem 5.2 is proved in Section 5.3. As a corollary of Theorems 5.1 and 5.2, in Section 5.4 we will also discuss some continuity in the $C^1$ topology of the Hausdorff dimension at conformal $C^{1+\kappa}$ constructions under the disjoint open set condition. For discussions of the constructions of self similar sets using similitudes and their dimensions, see Hutchinson (1981), Mauldin and Williams (1988). For the constructions using "conformal" contraction maps, see Bedford (1988). Other related works can be found in Dekking (1982), Bedford (1986) and Falconer (1990).

## §5.2. The Upper Bound

LEMMA 5.2.1. *If all construction diffeomorphisms $\varphi_{ij}$ are $C^{1+\kappa}$ and satisfy the $\kappa$-pinching condition, then for any $\varepsilon > 0$, there exists $\delta > 0$ depending only on $\varepsilon$, such that for all $x \in E$, all $a$ with $0 < a < \delta$, and all $\underline{x} = (x_0, x_1, \cdots)$ in $\Sigma$, all integers $n > 0$, we have*

$$\varphi_{x_0 \cdots x_n} B(x, a) \subset \varphi_{x_0 \cdots x_n}(x) + (1+\varepsilon)^n T_x \varphi_{x_0 \cdots x_n} B(0, a). \tag{5.2.1}$$

*Here $B(x, a)$ denotes a ball of radius $a$ centered at $x$ in $R^l$.*

PROOF: Using Taylor's formula, for any $y,\ w \in R^l$,

$$\varphi_{x_0 x_1}(y + w) = \varphi_{x_0 x_1}(y) + T_y \varphi_{x_0 x_1}(w) + r_{x_0 x_1}(w, y). \tag{5.2.2}$$

Since $E$ is compact, we can find some constants $C > 0$ and $c > 0$, such that for all $y \in E$ and $w \in R^l$ with $\|w\| \le c$, we have $\|r_{x_0 x_1}(w, y)\| < C\|w\|^{1+\kappa}$. We will set also $b = \min_{x \in E, i, j}\{\alpha_{l,ij}(x)\}$.

Fix any small $\varepsilon > 0$. Since $E$ is compact and all construction diffeomorphisms satisfy the $\kappa$ pinching condition, without loss of generality we can assume $\varepsilon$ to be so small that for all pairs $(i,j)$,

$$\|(1+\varepsilon)T_x\varphi_{ij}\| < 1 \text{ for all } x \in E; \tag{5.2.3}$$

$$(1+\varepsilon)^{\kappa}\alpha_{1,ij}(x)^{1+\kappa} < \alpha_{l,ij}(x) \text{ for all } x \in E. \tag{5.2.4}$$

Pick $\delta > 0$, with $\delta < \min\{c, (b\varepsilon/C)^{1/\kappa}\}$. Thus $\delta^{\kappa} < \varepsilon\alpha_{l,ij}(x)/C$, for all $x \in E$ and all pairs of $(i,j)$. Let $a \leq \delta$, and pick any $w \in R^l$ with $\|w\| < a$. For any $x$ in $E$, $\|r_{x_0x_1}(w,x)\| < C\|w\|^{1+\kappa} < Ca^{1+\kappa} \leq aC\delta^{\kappa} \leq a\varepsilon\alpha_{l,x_0x_1}(x)$, and thus $r_{x_0x_1}(w,x) \in \varepsilon\alpha_{l,x_0x_1}(x)B(0,a)$. Since $\varepsilon\alpha_{l,x_0x_1}(x)B(0,a) \subset \varepsilon T_x\varphi_{x_0x_1}B(0,a)$, it follows from (5.2.2) that

$$\begin{aligned}
\varphi_{x_0x_1}(x+w) &= \varphi_{x_0x_1}(x) + T_x\varphi_{x_0x_1}(w) + r_{x_0x_1}(w,x) \\
&\in \varphi_{x_0x_1}(x) + T_x\varphi_{x_0x_1}B(0,a) + \varepsilon T_x\varphi_{x_0x_1}B(0,a) \\
&= \varphi_{x_0x_1}(x) + (1+\varepsilon)T_x\varphi_{x_0x_1}B(0,a).
\end{aligned}$$

That gives (5.2.1) for $n = 1$. Now the induction hypothesis gives

$$\begin{aligned}
\varphi_{x_0x_1\cdots x_n}B(x,a) &= \varphi_{x_0x_1}\varphi_{x_1\cdots x_n}B(x,a) \\
&\subset \varphi_{x_0x_1}[\varphi_{x_1\cdots x_n}(x) + (1+\varepsilon)^{n-1}T_x\varphi_{x_1\cdots x_n}B(0,a)].
\end{aligned}$$

Using (5.2.2):

$$\begin{aligned}
&\varphi_{x_0x_1}[\varphi_{x_1\ldots x_n}(x) + (1+\varepsilon)^{n-1}T\varphi_{x_1\ldots x_n}(w)] \\
&= \varphi_{x_0\ldots x_n}(x) + (1+\varepsilon)^{n-1}T\varphi_{x_0\ldots x_n}(w) \\
&+ r_{x_0x_1}((1+\varepsilon)^{n-1}T\varphi_{x_1\ldots x_n}(w), \varphi_{x_1\ldots x_n}(x)).
\end{aligned}$$

Because of (5.2.3), $\|(1+\varepsilon)^{n-1}T\varphi_{x_1\ldots x_n}(w)\| < \|w\| < a$, where $w \in B(0,a)$. Using (5.2.4), we have

$$\begin{aligned}
&\|r_{x_0x_1}((1+\varepsilon)^{n-1}T\varphi_{x_1\ldots x_n}(w), \varphi_{x_1\ldots x_n}(x))\| \\
&< C\|(1+\varepsilon)^{n-1}T\varphi_{x_1\ldots x_n}(w)\|^{1+\kappa} \\
&\leq C(1+\varepsilon)^{(n-1)(1+\kappa)} \cdot [\alpha_{1,x_1x_2}(\varphi_{x_2\cdots x_n}(x))\ldots\alpha_{1,x_{n-1}x_n}(x)\|w\|]^{1+\kappa} \\
&< C(1+\varepsilon)^{n-1}\alpha_{l,x_1x_2}(\varphi_{x_2\cdots x_n}(x))\ldots\alpha_{l,x_{n-1}x_n}(x)\|w\|^{1+\kappa} \\
&< (1+\varepsilon)^{n-1}a^{\kappa}C\alpha_{l,x_1x_2}(\varphi_{x_2\cdots x_n}(x))\ldots\alpha_{l,x_{n-1}x_n}(x)\|w\| \\
&< \varepsilon(1+\varepsilon)^{n-1}\alpha_{l,x_0x_1}(\varphi_{x_1\cdots x_n}(x))\cdot\alpha_{l,x_1x_2}(\varphi_{x_2\cdots x_n}(x))\ldots\alpha_{l,x_{n-1}x_n}(x)a.
\end{aligned}$$

On the other hand $T_x\varphi_{ij}B(0,a) \supset \alpha_{l,ij}(x)B(0,a)$, and it follows that

$$T_x\varphi_{x_0\cdots x_n}B(0,a) \supset \alpha_{l,x_0x_1}(\varphi_{x_1\cdots x_n}(x))\cdots\alpha_{l,x_{n-1}x_n}(x)B(0,a).$$

Hence $r_{x_0x_1}((1+\varepsilon)^{n-1}T\varphi_{x_1\ldots x_n}(w), \varphi_{x_1\ldots x_n}(x)) \in \varepsilon(1+\varepsilon)^{n-1}T_x\varphi_{x_0\ldots x_n}B(0,a)$. Therefore,

$$\begin{aligned}
&\varphi_{x_0x_1}[\varphi_{x_1\ldots x_n}(x) + (1+\varepsilon)^{n-1}T\varphi_{x_1\ldots x_n}(w)] \\
&\in \varphi_{x_0\ldots x_n}(x) + (1+\varepsilon)^{n-1}T_x\varphi_{x_0\ldots x_n}B(0,a) + \varepsilon(1+\varepsilon)^{n-1}T_x\varphi_{x_0\ldots x_n}B(0,a) \\
&\subset \varphi_{x_0\ldots x_n}(x) + (1+\varepsilon)^{n}T_x\varphi_{x_0\ldots x_n}B(0,a).
\end{aligned}$$

Thus (5.2.1) is true for $n$. That completes the induction process. ∎

I have learned that Jiang (1991) has a distortion lemma for a regular non-conformal semigroup, which is a semigroup of pinched contracting diffeomorphisms. His version is stronger than our version here. However, for our purpose of estimating Hausdorff dimensions here, our version is strong enough.

**Proposition 5.2.2.** *If all construction diffeomorphisms $\varphi_{ij}$ are $C^{1+\kappa}$ and satisfy the $\kappa$-pinching condition where $0 < \kappa \leq 1$, and if the topological pressure $P(\sigma, \lambda_t) < 0$ where $\sigma$ is the shift map in $\Sigma$, then the Hausdorff dimension $HD(E) \leq t$.*

**Proof:** Choose small $\varepsilon > 0$ satisfying both (5.2.3) and (5.2.4), with

$$P(\sigma, \lambda_t) < -2t\varepsilon.$$

By Lemma 5.2.1, there exists $\delta > 0$ such that (5.2.1) holds for all integers $n > 0$ and each $x \in E$, when $0 < a < \delta$.

We fix $a < \delta$ small enough, and a positive integer $n$ big enough, such that (See Walters (1982) or Section 4.2 for notations)

$$\log P_n(\sigma, \lambda_t, a) < -2nt\varepsilon.$$

Recall that $\pi$ is Hölder continuous. Suppose that $\gamma$ is the exponent such that there exists a constant $D$ with

$$|\pi(\underline{x}) - \pi(\underline{y})| < D \cdot d(\underline{x}, \underline{y})^{\gamma}$$

for all $\underline{x}$, $\underline{y}$ in $\Sigma$. Fix $a' < \min\{D^{-1/\gamma} a^{1/\gamma}, a\}$. Pick $m$ with $2^{-m-1} < a' \leq 2^{-m}$. Let

$$K' = \{(x_0, \cdots, x_{m+n}) | \text{ there exists } \underline{x} \in \Sigma \text{ with } \underline{x} = (x_0, \cdots, x_{m+n}, \cdots)\}.$$

Choose for each word $(x_0, \cdots, x_{m+n})$ in $K'$ a point $\underline{x}$ in $\Sigma$ with the initial of $x_0, \ldots, x_{m+n}$, to form a subset $K$ of $\Sigma$. The subset $K$ is $(n, a')$ separated, and is

maximal in the sense that one cannot add another point to $K$ such that it is still $(n, a')$ separated. Thus, the collection $\{\sigma^{-n}B(\sigma^n\underline{x}, a')|\underline{x} \in K\}$ is an open cover for $\Sigma$. Notice that $\pi\underline{x} = \varphi_{x_0x_1}\pi\sigma\underline{x}$. Since $\pi B(\underline{x}, a') \subset B(\pi(\underline{x}), a)$ and

$$\pi\{\sigma^{-n}B(\sigma^n\underline{x}, a')|\underline{x} \in K\} \subset \{\varphi_{x_0x_1\cdots x_n}B(\pi\sigma^n\underline{x}, a)|\underline{x} = (x_0, x_1, \ldots, x_n \ldots) \in K\}$$

it follows that

$$\{\varphi_{x_0x_1\cdots x_n}B(\pi\sigma^n\underline{x}, a)|\underline{x} = (x_0, x_1, \ldots, x_n \ldots) \in K\}$$

is an open cover for $E = \cup_{i=1}^{l} E_i$. Also, using (5.2.1) of Lemma 5.2.1 we have

$$\varphi_{x_0\cdots x_n}B(\pi\sigma^n\underline{x}, a) \subset \varphi_{x_0\cdots x_n}(\pi\sigma^n\underline{x}) + (1+\varepsilon)^n T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n}B(0, a). \tag{5.2.5}$$

The right side of (5.2.5) is an ellipsoid with axes

$$\{a(1+\varepsilon)^n\alpha_j(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n})|1 \leq j \leq l\}.$$

Pick $j$ with $j-1 \leq t < j$. Then that ellipsoid can be covered by

$$\begin{aligned} C \cdot \alpha_1(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n}) \cdots \alpha_j(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n})/\alpha_j(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n})^j \\ = C \cdot \omega_{j-1}(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n})\alpha_j^{-j+1}(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n}) \end{aligned}$$

balls of radius $a(1+\varepsilon)^n\alpha_j(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n})$, where the constant $C > 0$ depends only on the dimension of $R^l$. Now we calculate the Hausdorff $t$-measure of $E$, using the smaller balls of radius $a(1+\varepsilon)^n \cdot \alpha_j(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n}) < a$ to cover the

open set $\varphi_{x_0\cdots x_n}B(\pi\sigma^n\underline{x}, a)$. If $\{P_i : i \in I\}$ is an open cover for $E$ where $P_i$ is a ball of radius $r_i$, then we define $|I| = \max_{i\in I} r_i$ and

$$\mathcal{H}_a^t = \inf_{|I|<a} \sum_{i\in I} r_i^t.$$

We have

$$\begin{aligned}
\mathcal{H}_a^t &\leq \sum_{\underline{x}\in K} C\omega_{j-1}(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n})\alpha_j^{-j+1}(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n})[a(1+\varepsilon)^n\alpha_j(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n})]^t \\
&= (1+\varepsilon)^{nt}a^tC\sum_{\underline{x}\in K}\omega_t(T_{\pi\sigma^n\underline{x}}\varphi_{x_0\cdots x_n}) \\
&\leq (1+\varepsilon)^{nt}C\sum_{\underline{x}\in K}\omega_t(T_{\pi\sigma\underline{x}}\varphi_{x_0x_1})\omega_t(T_{\pi\sigma^2\underline{x}}\varphi_{x_1x_2})\cdots\omega_t(T_{\pi\sigma^n\underline{x}}\varphi_{x_{n-1}x_n}) \\
&= (1+\varepsilon)^{nt}C\sum_{\underline{x}\in K}\exp[\lambda_t(\underline{x})+\lambda_t(\sigma\underline{x})+\cdots+\lambda_t(\sigma^{n-1}\underline{x})] \\
&\leq (1+\varepsilon)^{nt}CP_n(\sigma,\lambda_t,a') \\
&\leq (1+\varepsilon)^{nt}CP_n(\sigma,\lambda_t,a) \\
&\leq C\exp(nt\varepsilon)\exp(-2nt\varepsilon) \\
&\to 0,
\end{aligned}$$

as $n \to \infty$. Thus $\mathcal{H}_a^t = 0$. Since $a$ can be arbitrarily small, $\mu(t) = 0$. It follows that $HD(E) \leq t$. ∎

Proof of Theorem 5.1: $P(\sigma, \lambda_t)$ is a decreasing function of $t$'s since $E$ is compact and $\lambda_t$ is strictly decreasing with respect to $t$. So there is only one real number $t$ such that $P(\sigma, \lambda_t) = 0$. Also, the unique $t$ with $P(\sigma, \lambda_t) = 0$ is equal to

$$\inf\{t : P(\sigma, \lambda_t) < 0\}.$$

Consequently, we have $HD(E) \leq t$ where $P(\sigma, \lambda_t) = 0$. ∎

## §5.3. The Lower Bound

PROOF OF THEOREM 5.2: Notice that for each $t$, the map $\lambda_t$ is Hölder continuous on $\Sigma$. So there exists an equilibrium state $\mu$ for $\lambda_t$, in the sense that

$$P(\sigma, \lambda_t) = h_\mu(\sigma) + \int \lambda_t d\mu.$$

Fix any $\rho > 0$, let's estimate the $\mu$ measure of a ball $B(z, \rho)$ centered at $z$ with radius $\rho$. For each $\underline{x} \in \Sigma$ choose the unique $n = n(\underline{x}) \geq 0$ such that the diameters satisfy

$$\operatorname{diam}(U_n(\underline{x})) \leq \rho < \operatorname{diam}(U_{n-1}(\underline{x})).$$

LEMMA 5.3.1. *There exists a constant $c > 0$ such that for all $\underline{x} \in \Sigma$, the open set $U_{n(\underline{x})}(\underline{x})$ is contained in a ball of radius $\rho$ and contains a ball of radius $c\rho^{1+\kappa}$.*

PROOF: It is clear that $U_n(\underline{x})$ is contained in a ball of radius $\rho$. Since the radius of $U_n(\underline{x})$ decreases to 0 as $n$ grows to infinity, without loss of generality we can assume the maximum diameter $R$ of all $U_i$ is less than the number $\delta$ given in Lemma 5.2.1. Also pick $r$ small enough that each $U_i$ contains a ball of radius $r$. Then $U_n(\underline{x})$ contains a ball of radius

$$r \cdot \alpha_{l,x_0x_1}(\pi\sigma\underline{x}) \cdots \alpha_{l,x_{n-1}x_n}(\pi\sigma^n\underline{x}) > r \cdot \alpha_{1,x_0x_1}^{1+\kappa}(\pi\sigma\underline{x}) \cdots \alpha_{1,x_{n-1}x_n}^{1+\kappa}(\pi\sigma^n\underline{x}).$$

But on the other hand

$$\rho \leq \operatorname{diam}(U_{n-1}(\underline{x})) \leq \alpha_{1,x_0x_1}(\pi\sigma\underline{x}) \cdots \alpha_{1,x_{n-2}x_{n-1}}(\pi\sigma^{n-1}\underline{x})R,$$

which implies that

$$\alpha_{1,x_0x_1}(\pi\sigma\underline{x}) \cdots \alpha_{1,x_{n-1}x_n}(\pi\sigma^n\underline{x}) \geq \alpha_1\rho/R$$

where the constant $\alpha_1 = \min_{y \in E, i, j}\{\alpha_{1,ij}(y)\} > 0$ does not depend on either $n$ or $\underline{x}$. Therefore $U_n(\underline{x})$ contains a ball of radius $> r\rho^{1+\kappa}\alpha_1^{1+\kappa}/R^{1+\kappa}$. Writing $c = \alpha_1^{1+\kappa}r/R^{1+\kappa}$ a constant, then $U_n(\underline{x})$ contains a ball of radius $c\rho^{1+\kappa}$ as desired. ∎

For two points $\underline{x}$, $\underline{y} \in \Sigma$, since the construction maps satisfy the open set condition, $U_{n(\underline{x})}(\underline{x})$ and $U_{n(\underline{y})}(\underline{y})$ are either equal or disjoint. Let $\Gamma \subset \Sigma$ be a subset such that $\{U_{n(\underline{x})}(\underline{x}) | \underline{x} \in \Gamma\}$ is a disjoint collection which contains all $U_{n(\underline{x})}(\underline{x})$ for $\underline{x} \in \Sigma$. Notice that $\{\bar{U}_{n(\underline{x})}(\underline{x}) | \underline{x} \in \Gamma\}$ covers $E$.

LEMMA 5.3.2. *(Similar to Hutchinson (1981) 5.3 (a)) At most $3^l c^{-l} \rho^{-\kappa l}$ of $\{\bar{U}_{n(\underline{x})}(\underline{x}) | \underline{x} \in \Gamma\}$ can meet $B(z, \rho)$.*

PROOF: Suppose that $\bar{V}_1, \cdots, \bar{V}_m$ in $\{\bar{U}_{n(\underline{x})}(\underline{x}) | \underline{x} \in \Gamma\}$ meet $B(z, \rho)$. Then each of them is a subset of $B(z, 3\rho)$. By the definition of $\Gamma$ the sets in the collection $\{\bar{U}_{n(\underline{x})}(\underline{x}) | \underline{x} \in \Gamma\}$ are disjoint. Comparing the volumes we have

$$mJc^l \rho^{l(1+\kappa)} \leq J3^l \rho^l$$

where $J$ is the volume of a unit ball in $R^l$. Hence $m \leq 3^l c^{-l} \rho^{-\kappa l}$. ∎

Let

$$C_n(\underline{x}) = \{\underline{y} = (y_0, y_1, \cdots) \in \Sigma | y_0 = x_0, \cdots, y_n = x_n\}$$

be an $n$ cylinder. Recall that $\mu$ is a Gibbs measure (see Bowen (1975) for a discussion or Bedford (1988) for a summary). There exists a constant $d > 0$, with

$$\mu(C_n(\underline{x})) \in [d^{-1}, d] \cdot \exp(-P(\sigma, \lambda_t)n + S_n\lambda_t(\underline{x})),$$

for each cylinder $C_n(\underline{x})$ in $\Sigma$. Thus

$$\mu(C_n(\underline{x})) \in [d^{-1}, d] \cdot \exp(S_n\lambda_t(\underline{x})),$$

since $P(\sigma, \lambda_t) = 0$. So,

$$\begin{aligned}\mu(C_n(\underline{x})) &\le d \exp S_n \lambda_t(\underline{x}) \\ &\le d[\alpha_{1,x_0x_1}(\pi\sigma\underline{x}) \cdots \alpha_{1,x_{n-1}x_n}(\pi\sigma^n\underline{x})]^t \\ &\le d[\alpha_{l,x_0x_1}(\pi\sigma\underline{x}) \cdots \alpha_{l,x_{n-1}x_n}(\pi\sigma^n\underline{x})]^{t/(1+\kappa)} \\ &\le d \cdot [\text{ diam } (U_{n(\underline{x})})/r]^{t/(1+\kappa)}\end{aligned}$$

Hence if $n = n(\underline{x})$ we obtain

$$\mu(C_{n(\underline{x})}(\underline{x})) \le d\rho^{t/(1+\kappa)}/r^{t/(1+\kappa)}.$$

Noticing $\pi C_n(\underline{x}) \supset \bar{U}_n(\underline{x}) \cap E$, by Lemma 5.3.2,

$$\pi_*\mu(B(z,\rho)) \le [3^l c^{-l} d r^{-t/(1+\kappa)}]\rho^{t/(1+\kappa)-l\kappa}.$$

By the Frostman lemma (see Chapter 2 or Kahane (1985) for a proof),

$$HD(E) \ge \frac{t}{1+\kappa} - l\kappa.$$

That completes the estimate for the lower bound. ∎

## §5.4. Some Continuity in the $C^1$ Topology

The construction of the self similar set $E_\varphi$ depends on the contracting diffeomorphisms $\{\varphi_{ij}\}$. Now let us fix $0 < \beta \le 1$, and consider a $C^1$ perturbation to a $C^{1+\beta}$ conformal construction with diffeomorphisms $\{\varphi_{ij}\}$, and obtain another

matrix of contracting diffeomorphisms $\{\psi_{ij}\}$, which is not necessarily conformal. Denote the new self similar set for $\psi$ by $E_\psi$. Define

$$d_{C^1}(\varphi,\psi) = \max_{i,j} d_{C^1}\{(\varphi_{ij},\psi_{ij})\},$$

where the later $d_{C^1}$ is the $C^1$ metric. Note that for any $\kappa < \beta$, when $\psi$ is sufficiently $C^1$ close to $\varphi$, $\psi$ must be $C^{1+\kappa}$ and also $\kappa$ pinched. The following theorem is a corollary of Theorems 5.1 and 5.2. It tells that at a $C^{1+\beta}$ conformal construction satisfying the open set condition for self similar sets, the Hausdorff dimension $HD(E_\psi)$ depends continuously on $\{\psi_{ij}\}$ in $C^1$ topology.

THEOREM 5.4.1. *Let $\{\varphi_{ij}\}$ be a matrix of $C^{1+\beta}$ conformal construction diffeomorphisms for the self similar set $E_\varphi$, satisfying the open set condition. For any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $C^{1+\beta}$ construction $\psi$ satisfying the open set condition, with $d_{C^1}(\varphi,\psi) < \delta$, we have*

$$|HD(E_\varphi) - HD(E_\psi)| < \varepsilon.$$

PROOF: Let

$$\lambda_{\varphi,s}(\underline{x}) = \log \omega_s(T\varphi_{x_0x_1}(\pi\sigma\underline{x})),$$

$$\lambda_{\psi,s}(\underline{x}) = \log \omega_s(T\psi_{x_0x_1}(\pi\sigma\underline{x})),$$

be two real functions on $\Sigma$ as defined in Definition 5.1.1. Let $t$ be such that $P(\sigma,\lambda_{\varphi,t}) = 0$. Because $\varphi_{ij}$'s are conformal, the Hausdorff dimension of $E_\varphi$ equals $t$. Also, remark that $P(\sigma,\lambda_{\varphi,t+\varepsilon}) < 0$ for any $\varepsilon > 0$.

Now fix any $\varepsilon > 0$. Let $\kappa = \min\{\beta, \varepsilon/4l\}$ and let

$$\varepsilon' = \frac{1}{2}\min\{-P(\sigma,\lambda_{\varphi,t+\varepsilon}), P(\sigma,\lambda_{\varphi,t-\varepsilon/4})\} > 0. \tag{5.4.1}$$

Since $\varphi$ is $C^{1+\beta}$ and conformal, there is $\delta > 0$ such that the $C^{1+\beta}$ diffeomorphism $\psi$ is $C^{1+\kappa}$ and $\kappa$ pinched with $|\lambda_{\varphi,s}(\underline{x}) - \lambda_{\psi,s}(\underline{x})| < \varepsilon'$ for all $s \in [0, l]$, if $d_{C^1}(\varphi, \psi) < \delta$. Then

$$P(\sigma, \lambda_{\psi,t+\varepsilon}) < P(\sigma, \lambda_{\varphi,t+\varepsilon} + \varepsilon') \leq P(\sigma, \lambda_{\varphi,t+\varepsilon}) + \varepsilon' < 0.$$

So

$$HD(E_\psi) \leq t + \varepsilon = HD(E_\varphi) + \varepsilon,$$

by Proposition 5.2.2.

On the other hand, by (5.4.1), when $d_{C^1}(\varphi, \psi) < \delta$, we have

$$P(\sigma, \lambda_{\psi,t-\varepsilon/4}) > P(\sigma, \lambda_{\varphi,t-\varepsilon/4} - \varepsilon') \geq P(\sigma, \lambda_{\varphi,t-\varepsilon/4}) - \varepsilon' > 0.$$

So we have some $s \geq t - \varepsilon/4$ with $P(\sigma, \lambda_{\psi,s}) = 0$ since $P(\sigma, \lambda_{\psi,s})$ is strictly decreasing with respect to $s$. By Theorem 5.2,

$$\begin{aligned} HD(E_\psi) &\geq s/(1+\kappa) - l\kappa \geq s/(1 + \varepsilon/4l) - l(\varepsilon/4l) \\ &> s(1 - \varepsilon/4l) - \varepsilon/4 \geq s - \varepsilon/2 > t - \varepsilon. \end{aligned}$$

It then follows that

$$HD(E_\psi) \geq t - \varepsilon = HD(E_\varphi) - \varepsilon,$$

as desired. ∎

We say a construction $\varphi$ with diffeomorphisms $\{\varphi_{ij}\}$ satisfies the strong open set condition if there are open sets $U_1, \cdots, U_l$ in $R^l$ with $\varphi_{ij}(\bar{U}_j) \subset U_i$ for all $i, j$. If the construction $\varphi$ satisfies the strong open set condition, then $\psi$ must also satisfy the strong open set condition if it is $C^1$ close enough to $\varphi$. Thus we have obtained an immediate corollary of the above Theorem 5.4.1:

COROLLARY 5.4.2. *Let $\{\varphi_{ij}\}$ be a matrix of $C^{1+\beta}$ conformal construction diffeomorphisms for the self similar set $E_\varphi$, satisfying the strong open set condition. For any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $C^{1+\beta}$ construction $\psi$ with $d_{C^1}(\varphi, \psi) < \delta$, we have*

$$|HD(E_\varphi) - HD(E_\psi)| < \varepsilon.$$

Finally we have a remark on the continuity of the Hausdorff dimension in the $C^1$ topology at non-conformal constructions.

REMARK 5.4.3: The following example shows that if the "conformal" condition for the construction diffeomorphisms $\{\varphi_{ij}\}$ fails, then the results in Theorem 5.4.1 and Corollary 5.4.2 can be false. The example is derived from Example 9.10 of Falconer (1990), pp 127-128.

Let $S, T_\lambda : R^2 \to R^2$ be given by

$$S(x,y) = (x/2, y/3 + 2/3), \quad T_\lambda(x,y) = (x/2 + \lambda, y/3)$$

where $\lambda \in [0, 1/2)$ and $(x,y) \in R^2$. Let $\varphi_{11} = \varphi_{21} = S$, $\varphi_{12} = \varphi_{22} = T_0$. Take $\psi_\lambda = \{\psi_{ij,\lambda}\}$ where $\psi_{11,\lambda} = \psi_{21,\lambda} = S$, and $\psi_{12,\lambda} = \psi_{22,\lambda} = T_\lambda$. The strong open set condition is met for $\{\varphi_{ij}\}$. In fact if we let $U_1 = U_2 = (-1/8, 9/8)^2 \subset R^2$ then $\varphi_{ij}(\bar{U}_j) \subset U_i$.

Let $E_\varphi, E_{\psi_\lambda}$ be the self similar sets for $\varphi$ and $\psi_\lambda$. Considering the projection of $E_{\psi_\lambda}$ to the $x$-axis, one knows that $HD(E_{\psi_\lambda}) \geq 1$ for $\lambda > 0$. But $E_\varphi$ is a Cantor set contained in the $y$-axis with the Hausdorff dimension

$$HD(E_\varphi) = \log 2 / \log 3 < 1.$$

Since $d_{C^1}(\psi_\lambda, \varphi) = \lambda$, so letting $\lambda \to 0$ we know the Hausdorff dimension is not continuous at $\varphi$. We notice that $\{\varphi_{ij}\}$ are not conformal although $\{\varphi_{ij}\}$ and $\{\psi_{ij,\lambda}\}$ are all 3/4 pinched.

CHAPTER 6

# THE HÉNON ATTRACTOR

In the 1970's Hénon et al. conducted numerical experiments on the following quadratic automorphisms of the plane, $H : R^2 \to R^2$ defined by

$$H(X, Y) = (1 + Y - aX^2, bX), \text{ where } a, b \text{ are parameters} \quad ,$$

which "show" a "strange" attractor. We now call this map a Hénon map. See Hénon (1976), Hénon and Pomeau (1975) for numerical results. Say $\Lambda$ is an attractor of $H$ if it is invariant under $H$ and there is a neighborhood $U$ of $\Lambda$ such that every point $z \in U$ has $\lim_{n\to\infty} d(H^n(z), \Lambda) = 0$. A Hénon map is regarded as the simplest smooth example that computer simulations indicate the existence of a strange attractor. Since Hénon's numerical experiments a number of authors have studied the dynamical behavior of the Hénon mappings. Devaney (1986) gives an introduction to this subject. Milnor (1988) noticed many Hénon mappings are non-expansive, and the topological entropies of the Hénon mappings fall within $[0, \log 2]$. In this chapter, we give an upper bound for the capacity and the Hausdorff dimension of a Hénon attractor.

Under the coordinate change suggested Devaney and Nitecki (1979), $X = x/a$, $Y = by/a$ we have the following convenient form of the Hénon mapping:

$$H(x, y) = (a + by - x^2, x), \text{ where } a, b \text{ are parameters.}$$

Benedicks and Carleson (1991) shows the following: Let $W^u = W^u(z, H)$ be the unstable manifold of $H$ at the fixed point $z$ of $H$ in the first quadrant. Then there is a $b_0 > 0$ such that for all $b \in (0, b_0)$ there is a set $E_b$ of positive one dimensional Lebesgue measure such that for all $a \in E_b$:

(1)There is an open set $U = U(a, b)$ such that for all $z \in U$,

$$d(H^n(z), \bar{W^u}) \to 0, \text{ as } n \to \infty.$$

(2)There is a point $z_0 = z_0(a, b) \in W^u$ such that $\{H^n(z_0)\}_{n=0}^{\infty}$ is dense in $W^u$.

We consider the case when $b_0 < \frac{1}{2}$. Also, it is clear that $W^u$ is $C^\infty$ and one dimensional. Thus

$$\bar{C}(\Lambda) \geq \underset{\_}{C}(\Lambda) \geq HD(\Lambda) \geq 1.$$

Now let $R$ be the bound for $\Lambda$ such that for every $(x, y) \in \Lambda$ we have $|x|, |y| \leq R$, and denote

$$z_n = (x_n, y_n) = H^n(z_0), \quad m = \frac{2R+1}{b},$$

and

$$A = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(b + \frac{2|x_i|}{m} + \frac{1}{m^2})$$

Since

$$b + \frac{2|x_i|}{m} + \frac{1}{m^2} \leq 2b < 1,$$

we have $A < 0$. For the capacity of an attractor, we shall prove the following:

THEOREM IV. *If $\Lambda$ is a compact invariant attractor under the Hénon map $H$, as proved existing by Benedicks and Carleson (1991) then the upper capacity*

$$\bar{C}(\Lambda) \leq 2 + \frac{A}{\log m} < 2.$$

Douady and Oesterlé have an upper bound for the Hausdorff dimension of attractors. Since $HD(\Lambda) \leq \bar{C}(\Lambda)$, our theorem IV also gives an upper bound for the Hausdorff dimension. In the case of a Hénon attractor our estimate is better than Douady and Oesterlé's result applied to a Hénon attractor.

Let $R_1$ be the larger root of the equation $x^2 - (1 + |b|)x - a = 0$. Let $S = [R_1, R_1] \times [R_1, R_1]$. Then any point outside $S$ on the plane will flee to infinity under either the Hénon map $H$ or $H^{-1}$, see Devaney (1986). In other words, any invariant set $\Lambda$ is contained in $S$, and thus bounded.

PROOF OF THEOREM IV: First take a look at the image of a ball under the Hénon map. Let $\|(x,y)\| = \max\{x,y\}$ and $d$ be the induced metric. After a coordinate change $\bar{x} = x/m, \bar{y} = y$ the Hénon map has the following form:

$$H(\bar{x},\bar{y}) = (ma + mb\bar{y} - \frac{1}{m}\bar{x}^2, \frac{\bar{x}}{m}).$$

Let us denote the new coordinate system also by $(x,y)$ from now on.

LEMMA 6.1. *Let*

$$B = \{(x,y) : |x-u| \leq r, |y-v| \leq r\}$$

*be a ball centered at $(u,v)$ of radius $r$ in our metric. Then $H(B)$ is contained in the following rectangle:*

$$[u_1 - l, u_1 + l] \times [v_1 - \frac{r}{m}, v_1 + \frac{r}{m}]$$

*where*

$$(u_1, v_1) = H(u,v), \text{ and } l = mbr + 2\frac{r|u|}{m} + \frac{r^2}{m}.$$

PROOF OF LEMMA 6.1: Since $B$ is bounded by $x = u \pm r, y = v \pm r$, the image of $B$, $H(B)$ will be bounded by

$$y = \frac{u}{m} \pm \frac{r}{m}, \text{ and } x = ma + mbv - my^2 \pm mbr.$$

Pick any point $(x, y)$ from the parabola

$$x = ma + mbv - my^2 - mbr \text{ with } |y - \frac{u}{m}| \leq \frac{r}{m}.$$

Then

$$|x - u_1| \leq mbr + 2\frac{r|u|}{m} + \frac{r^2}{m},$$

which completes the proof for the lemma.

Therefore, $H(B)$ can be covered by at most

$$1 + (mbr + 2\frac{r|u|}{m} + \frac{r^2}{m})/\frac{r}{m} = m^2b + 2|u| + r + 1$$

balls of radius $r/m$. Now let $\{B_i : i = 1, \cdots, k\}$ be a finite cover of $\Lambda$ where every ball $B_i$ has a radius less than $r/2$. Since $\{z_n : n = 0, 1, \cdots,\}$ is dense in $\Lambda$, one can pick some $n_i$ such that $z_{n_i} \in B_i$. Then $\{B(z_{n_i}, r) : i = 1, \cdots, k\}$ is a cover of $\Lambda$. Let us write in order: $n_1 < n_2 < \cdots < n_k$. Using Lemma, if a ball $B$ centered at $(u, v)$ has radius $r$ and we write $H^i(u, v) = (u_i, v_i)$ then $H^n(B)$ can be covered by at most

$$\prod_{i=0}^{n-1} (m^2b + 2|u_i| + r + 1)$$

balls of radius $\frac{r}{m^n}$. Since $\{B(z_{n_i}, r) : i = 1, \cdots, k\}$ is a cover of $\Lambda$, the image under the diffeomorphism $H^n$,

$$\{H^n B(z_{n_i}, r) : i = 1, \cdots, k\}$$

is also a cover of $\Lambda$. Suppose $N = \max\{n_i : i = 1, \cdots, k\}$. Notice that $H^n B(z_{n_i}, r)$ is covered by at most

$$\prod_{j=i}^{i+n-1} (m^2 b + 2|x_{n_j}| + r + 1)$$

balls of radius $\dfrac{r}{m^n}$. But

$$\begin{aligned}\prod_{j=i}^{i+n-1} (m^2 b + 2|x_{n_j}| + r + 1) &\leq \prod_{j=0}^{n_i+n-1} (m^2 b + 2|x_j| + r + 1) \\ &\leq \prod_{j=0}^{N+n-1} (m^2 b + 2|x_j| + r + 1).\end{aligned}$$

Therefore the number of balls of radius $\dfrac{r}{m^n}$ needed to cover $\Lambda$ is at most

$$k \cdot \prod_{j=0}^{N+n-1} (m^2 b + 2|x_j| + r + 1).$$

By Lemma 1.3.1, we have

$$\begin{aligned}\bar{C}(\Lambda) &\leq \limsup_{n\to\infty} \frac{\log k \prod_{j=0}^{N+n-1} (m^2 b + 2|x_j| + r + 1)}{\log(r/m^n)} \\ &= \limsup_{n\to\infty} \frac{\log k + \sum_{j=0}^{N+n-1} \log(m^2 b + 2|x_j| + r + 1)}{n \log m + \log r} \\ &= \limsup_{n\to\infty} \frac{\sum_{j=0}^{N+n-1} \log(m^2 b + 2|x_j| + r + 1)}{n \log m} \\ &= \frac{1}{\log m} \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{N+n-1} \log(m^2 b + 2|x_j| + r + 1).\end{aligned}$$

Since $|x_i| \leq R$ for all $i$, and the function log is differentiable, there is a constant $C$ such that for all $r$ with $0 \leq r \leq 1$ and for all $j \geq 0$,

$$\begin{aligned}\log(m^2 b + 2|x_j| + 1) &\leq \log(m^2 b + 2|x_j| + r + 1)\\ &\leq \log(m^2 b + 2|x_j| + 1) + Cr.\end{aligned}$$

This implies by letting $r \to 0$ that

$$\begin{aligned}\bar{C}(\Lambda) &\leq \frac{1}{\log m} \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{N+n-1} \log(m^2 b + 2|x_j| + 1)\\ &= 2 + \frac{1}{\log m} \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{N+n-1} \log(b + \frac{2m|x_j| + 1}{m^2})\\ &= 2 + \frac{1}{\log m} \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log(b + \frac{2m|x_j| + 1}{m^2})\\ &= 2 + \frac{A}{\log m}.\end{aligned}$$

That completes the proof. ∎

# CHAPTER 7

## SUMMARY AND CONCLUSIONS

In this paper we are mainly concerned with the Hausdorff dimension of hyperbolic sets, the self similar sets and the Hénon attractors. These are all invariant sets under some smooth diffeomorphisms.

For a hyperbolic set of a diffeomorphism or a flow of diffeomorphisms, the uniform stable and unstable Lyapunov exponents have been defined. Using the uniform Lyapunov exponents we define a characteristic function. At some real positive number $t$ the characteristic function equals the value of the topological entropy of the diffeomorphism on the hyperbolic set. We prove that $t$ is an upper bound for the Hausdorff dimension of this hyperbolic set. Technically, we use a pinching condition to cope with the nonlinearity of the diffeomorphism, to prove that the iterated image of a ball is somehow contained in the image of the ball under the derivative of the iterated diffeomorphism. Some earlier results are thus improved.

In Chapter 4 we have also given upper bounds for the Hausdorff dimension of the transverse of a hyperbolic set with stable and unstable manifolds, using the topological pressure on the hyperbolic set.

Several authors have studied the self similar sets of iterated mapping systems in the cases where construction diffeomorphisms are conformal. In chapter 5 we study the case where the construction diffeomorphisms are not necessarily conformal. We give a new distortion lemma. Using topological pressure on a

shift space we give an upper estimate of the Hausdorff dimension, when the construction diffeomorphisms are $C^{1+\kappa}$ and satisfy a $\kappa$ pinching condition for some $\kappa \leq 1$. Moreover, if the construction diffeomorphisms also satisfy the disjoint open set condition we then give a lower bound for the Hausdorff dimension. We believe that the characteristic function $\lambda$ can be revised to achieve an even better estimate for the Hausdorff dimension. We also obtain the continuity of the Hausdorff dimension in the $C^1$ topology at conformal constructions.

In chapter 6 we study the image of a disc under the iteration of a Hénon map, which enables us to find an upper bound for the Hausdorff dimension and capacity of the Hénon attractor. That improves some earlier estimates when applied to the case of a Hénon attractor. Milnor (1988) noticed that many Hénon mappings are non-expansive, and the topological entropies of the Hénon mappings fall within $[0, \log 2]$. Since we cannot establish a distortion lemma similar to Lemma 3.4.2 for Hénon maps, it is unknown to us whether the Hausdorff dimension of a Hénon attractor can be related to its topological entropy. It is still an open problem.

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